Helicity Effects on Quantum Scattering

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1 Abstract

In this study, asymmetries in the collisions between bound states with like and unlike spin states are investigated. The goal is to provide theoretical ground for probing the angular momentum structure of the proton using helicity asymmetries in polarized proton-proton collisions. In order to provide proof of principle for the presence of asymmetries between like and unlike helicity collisions, this study uses nonrelativistic time dependent scattering theory. A simple collision model between two bound states is used with potentials specially selected for this study. While at first order there is no helicity asymmetry between like and unlike helicity collisions, terms at second order do not appear to vanish. However, further investigation is necessary to prove that these second order terms do not vanish unexpectedly.

2 Motivation

It is known that the spin of the proton is not entirely due to the spins of the constituent quarks [1]. One possible source for this missing spin is the orbital angular momentum of the constituents. In order to determine if there is any spin contribution from orbital angular momentum of the constituents, there is a need for experimental techniques that probe the angular momentum of the partons.

One experimental probe of the orbital angular momentum has been suggested and studied by Douglas Fields and his group at the University of New Mexico [2]. Fields has suggested that orbital angular momentum might have an effect on the observed momenta perpendicular to the collision axis of the final products of the hard scattering of two polarized protons. This momentum perpendicular to the collision axis is called transverse momentum. The effect of angular momentum on the final products would have on the transverse momentum depends on the impact parameter of the collision, as well as the relative directions of the angular momentum. Figure 1 illustrates this process.

Unfortunately, little progress has been made in determining the impact parameter of proton-proton collisions experimentally, so there is a question as to whether an effect can be seen when all impact parameters are summed over. A paper by Meng et al [3], making a similar suggestion for probing angular momentum using polarized proton-proton collisions, found that an effect could still be seen even after summing over all impact parameters. A more recent paper [4] by a member of Fields' group here at UNM considered semi-classical collisions of two rotating bodies, and also found that an effect should be seen after integrating over all impact parameters.

The effect that should be observed is an asymmetry in the transverse momentum to the beam axis between collisions with like helicity and collisions with unlike helicity. When a system has angular momentum in the same direction as it travels it has positive helicity, and when the angular

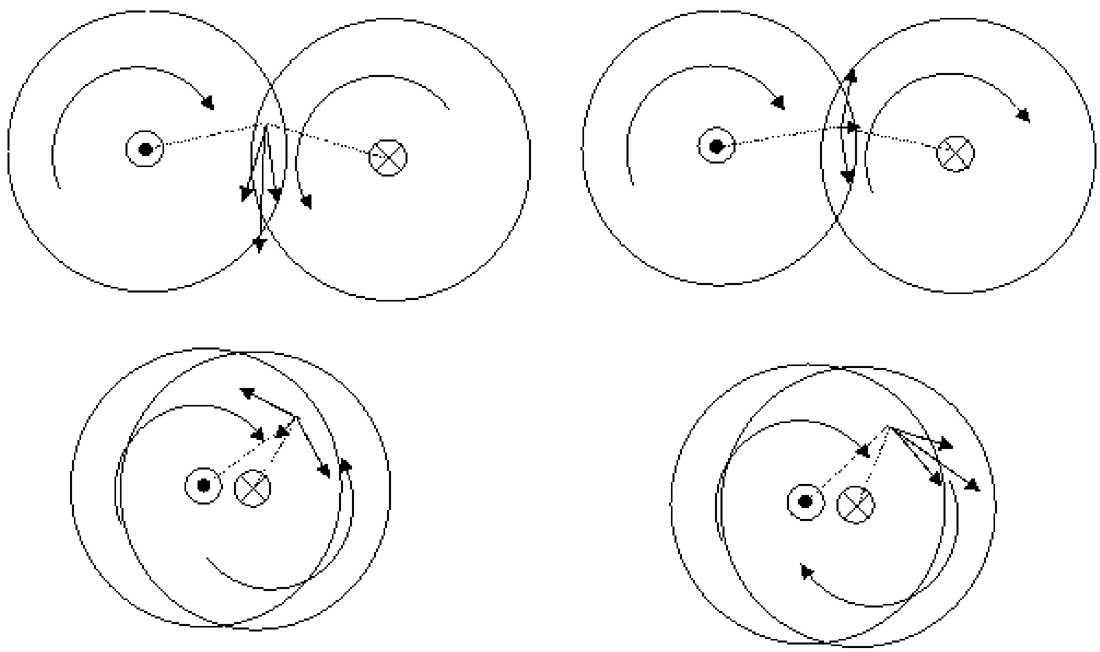

Figure 1: Diagrams for the classical collisions of rotating bodies with various impact parameters and helicity combinations, illustrating the transverse momentum of the collision products. Starting from upper left and moving clockwise, a like helicity collision with large impact parameter gives a large transverse momentum, an unlike helicity collision with large impact parameter gives a small transverse momentum, an unlike helicity collision with a small impact parameter gives a large transverse momentum, and a like helicity collision with a small impact parameter gives a small transverse momentum.

momentum is anti-aligned with its motion it has negative helicity. In like helicity collisions, both bodies have helicities of the same sign, while in unlike helicity collisions they have opposite signs.

Despite the amount of effort put into determining this effect using classical methods, there has been little work using quantum mechanical scattering theory to see if the effect should be expected. The purpose of this paper is to investigate helicity asymmetries in the quantum mechanical scattering of bound states with angular momentum. The next section is a brief introduction to time dependent quantum scattering, which will serve to prepare a reader unfamiliar with the subject for the calculations that follow. The next section will also introduce the notation used in this paper.

3 An Overview of Scattering Theory

Classical scattering theory involves the study of collisions. Quantum scattering theory is the extension of scattering theory into the quantum realm, and many of the features of classical scattering carry over. Of key importance to the theory of quantum scattering is the scattering operator, which will be discussed first. We begin by considering an elastic collision off some fixed potential, and generalize to inelastic and multiple particle scattering later. Throughout we will follow the formalism presented by Taylor [5].

3.1 The Scattering Operator

In a beginning discussion of classical scattering we are concerned with a particle with some given momentum, which scatters off of a fixed target. The scattering particle approaches the target from a long way off during a period of time which is very large compared to the time elapsed in the scattering event, and then with some new momentum travels a large distance over a long period of time before it is detected. Since the starting and ending distances are large, and the times elapsed before and after the collision are large, we consider that the particle has approached from spatial infinity at time $t = -\infty$ and afterward travels off to spatial infinity at time $t = +\infty$. We call the trajectory in from spatial infinity the in asymptote and the trajectory back out the out asymptote. Provided that whatever forces cause the scattering fall off quickly with distance from the target, we then treat the particle as a free particle along each of this asymptotes, which are then connected in the middle by some possibly complicated scattering trajectory.

In order to move this picture into the quantum realm, we imagine that there is some scattering state $|\psi\rangle$ at $t = 0$. At this time the state evolves under the full Hamiltonian H of the system. We then label the state at $t = -\infty$ as $|\psi_{in}\rangle$ and the state at $t = +\infty$ as $|\psi_{out}\rangle$. In each of these states the particle is a free particle evolving under the free Hamiltonian $H_0 = \frac{p^2}{2m}$.

In order to guarantee the validity of this picture, and also derive the results that follow, we make some assumptions about the scattering potential. We assume that the potential falls off faster than r^{-3} as r goes to infinity, and that the potential has a weaker singularity as r goes to zero than r^{-2}. We also assume that the potential has only a finite number of finite discontinuities. These requirements are fairly stringent; for example, these conditions exclude the Coulomb potential from consideration. It is worth noting that the Yukawa potential of nuclear interactions does indeed meet these requirements.

Given these assumptions, the picture of in and out asymptote free states holds, and we write

$$U(t)|\psi\rangle \quad \underset{t\to-\infty}{\longrightarrow} \quad U^0(t)|\psi_{in}\rangle \ , \tag{1}$$

$$U(t)|\psi\rangle \quad \underset{t\to+\infty}{\longrightarrow} \quad U^0(t)|\psi_{out}\rangle \ , \tag{2}$$

where

$$U(t) = e^{-iHt/\hbar} , \tag{3}$$
$$U^0(t) = e^{-iH^0t/\hbar} , \tag{4}$$

are the usual propogator and the free particle propogator, respectively. Since these are unitary operators (for time independent Hamiltonians, an assumption we make throughout), we act on both sides of these rough relations with $U^\dagger(t)$. We can then write that

$$|\psi\rangle = \lim_{t\to-\infty} U^\dagger(t)U^0(t)|\psi_{in}\rangle \equiv \Omega_+|\psi_{in}\rangle \tag{5}$$
$$|\psi\rangle = \lim_{t\to+\infty} U^\dagger(t)U^0(t)|\psi_{out}\rangle \equiv \Omega_-|\psi_{out}\rangle. \tag{6}$$

That these limits exist must be proved, and once they are they can be taken as the definition of the operators $\Omega_\pm$, which are known as the Møller operators. They carry the in asymptote forward in time to the scattering state, in the case of the Ω_+, and the out asymptote backward in time to the scattering state, in the case of the Ω_-. The proof of the existence of the above equations is somewhat tricky, and the interested reader is referred to Taylor. Since these operators carry the asymptote states onto only those scattering states where the particle can have come from infinity or escape to infinity, their range is orthogonal to the bound states of the Hamiltonian at $t = 0$.

As limits of unitary operators, these Møller operators are what is known isometric operators. They are not unitary since they map from a complete Hilbert space at $t = \pm\infty$ onto only part of the Hilbert space at $t = 0$. Nevertheless, in order to conserve probability, the $\Omega_\pm$ still satisfy $\Omega_\pm^\dagger\Omega_\pm = 1$, so we can write the out asymptote as a function of the scattering state,

$$\Omega_-^\dagger|\psi\rangle = \Omega_-^\dagger\Omega_-|\psi_{out}\rangle \tag{7}$$
$$= |\psi_{out}\rangle . \tag{8}$$

With this we can write

$$|\psi_{out}\rangle = \Omega_-^\dagger|\psi\rangle \tag{9}$$
$$= \Omega_-^\dagger\Omega_+|\psi_{in}\rangle \tag{10}$$
$$\equiv S|\psi_{in}\rangle, \tag{11}$$

where S is the scattering operator. The scattering operator evolves $|\psi_{in}\rangle$ to $|\psi_{out}\rangle$ and tells us everything about the scattering process in question. This formalism is very practical from an experimental standpoint. The state sent into the scattering process is in principle well controlled, and the state eventually emerging from the collision and entering into our detectors and can be measured precisely. Since the scattering operator is determined by the particular interaction under question, by knowing the in and out states experimenters can probe the interaction occurring in between.

We imagine some specific initial state $|\psi_{in}\rangle$ that evolves to some $|\psi+\rangle$ at $t = 0$ via the Ω_+ operator, and some specific final state $|\psi_{out}\rangle$ at $t = 0$ that is carried to some $|\psi-\rangle$ via Ω_-. The probability that this initial state evolves into this final state is then the squared product of the two at $t = 0$,

$$\begin{aligned} P &= |\langle\psi - |\psi+\rangle|^2 & (12) \\ &= |\langle\psi_{out}|\Omega_-^\dagger\Omega_+|\psi_{in}\rangle|^2 & (13) \\ &= |\langle\psi_{out}|S|\psi_{in}\rangle|^2 \; . & (14) \end{aligned}$$

As our final result regarding the scattering operator, it can be shown that the S operator conserves energy, and must contain the sum of the identity operator (particle is not scattered) and some element determining the scattering process. Altogether this yields the matrix element in the momentum representation,

$$\langle\mathbf{p}'|S|\mathbf{p}\rangle = \delta(\mathbf{p}' - \mathbf{p}) - 2\pi i\delta(E_{p'} - E_p)\langle\mathbf{p}'|T|\mathbf{p}\rangle \; , \tag{15}$$

where $\langle\mathbf{p}'|T|\mathbf{p}\rangle$ is called the on shell T matrix element and is sometimes just called $t(\mathbf{p}' \leftarrow \mathbf{p})$. "On shell" refers to the conservation of energy; the T operator is also extended to the "off shell" region where it is useful for mathematical analysis, but not physical in the sense that the overall scattering process must conserve energy. It should be noted that in relativistic scattering, very short lived processes occur off shell because of the uncertainty relationship $\Delta E\Delta t \geq \hbar/2$; this is the case for scattering off of virtual particles.

3.2 The Differential Cross Section

We imagine sending a beam of particles with random impact parameters at a target, and we can count the number of particles that are scattered. With the density of the beam measured in particles per unit area, n, the number of particles scattered is

$$N = \sigma n \; . \tag{16}$$

We see that σ is the probability of scattering, per particle per unit area. σ is of course the cross section, and is the effective cross sectional area of the target in the case where all particles entering the cross sectional area are scattered.

While σ represents the total cross section for all kinds of scattering, if we select out a specific scattering process, we have a different effective area for scattering. If we choose to examine the number of particles that scatter into a specific differential element of solid angle, $d\Omega$ then we arrive at the differential cross section for that scattering,

$$\sigma(d\Omega) = \frac{d\sigma}{d\Omega}d\Omega \; . \tag{17}$$

The differential cross section can then be integrated over some region of solid angle to arrive at the effective cross sectional area of the target for scattering into that region. Since the differential cross section is related to the probability of scattering, it seems clear that we can relate it to the scattering operator. This is accomplished by first recalling that the probability density of measuring some specific scattered final state in momentum space is

$$\begin{aligned} \rho(\mathbf{p}) &= \langle\psi_{out}|\mathbf{p}\rangle\langle\mathbf{p}|\psi_{out}\rangle & (18) \\ &= |\psi_{out}(\mathbf{p})|^2 \; , & (19) \end{aligned}$$

and then integrating this over a small element of solid angle $d\Omega$ and all magnitudes of momenta. This gives the probability of measuring some final state in a differential unit of solid angle,

$$P(d\Omega \leftarrow \psi_{in}) \quad = \quad d\Omega \int_0^\infty p^2 dp |\psi_{out}(\mathbf{p})|^2 \ . \tag{20}$$

It is worth noting here that we are using momentum space rather than a position representation. This may be a bit confusing, since when we speak of scattering into some unit of solid angle we usually picture some part of a sphere around the scattering event, clearly an element of position space. However, the unit of solid angle in momentum space gives the direction of the particle's momentum, and since it will travel in the direction of its momentum, we expect the particle to impact our detectors in a corresponding unit of solid angle in position space.

Next we recall that $|\psi_{out}\rangle = S|\psi_{in}\rangle$, and write the momentum representation wavefunction of the out asymptote in terms of the scattering operator and the in asymptote wavefunction,

$$\psi_{out}(\mathbf{p}) \quad = \quad \langle \mathbf{p}|\psi_{out}\rangle \tag{21}$$

$$= \quad \langle \mathbf{p}|S|\psi_{in}\rangle \tag{22}$$

$$= \quad \int d\mathbf{p'} \langle \mathbf{p}|S|\mathbf{p'}\rangle \psi_{in}(\mathbf{p'}) \ . \tag{23}$$

To finally arrive at the cross section, we need a beam of particles, each with a random impact parameter. We imagine then that each ψ_{in} is composed of a wavefunction $\phi(\mathbf{p})$ well peaked in momentum space, which is acted on by a rigid translation in position perpendicular to the beam. This translation by an impact parameter $\mathbf{b}$ in the momentum representation is achieved by multiplying $\phi(\mathbf{p})$ by $e^{-i\mathbf{b}\cdot\mathbf{p}/\hbar}$. These impact parameters are uniformly distributed over all possible values of b. Thus,

$$\psi_{in}(\mathbf{p}, \mathbf{b}) \quad = \quad \phi(\mathbf{p}) \exp^{-i\mathbf{b}\cdot\mathbf{p}/\hbar} \ . \tag{24}$$

We then need to integrate the probability of scattering over all impact parameters, so that

$$\frac{d\sigma}{d\Omega} d\Omega \quad = \quad \int d^2 b P(d\Omega \leftarrow \psi_{in}) \ . \tag{25}$$

Combining equations (20) - (25), and inserting equation (15) for the representation of the scattering operator, we have a mess. This mess can be resolved very nicely with the help of the assumption that $\phi(\mathbf{p})$ is well peaked about some momentum $\mathbf{p_0}$. We also have to avoid consideration of the flux into the small region $\mathbf{p_0}$, because in this region scattered particles are indistinguishable from unscattered particles. With these assumptions, the delta functions can be resolved to remove all the integrals, and the final result is that

$$\frac{d\sigma}{d\Omega} \quad = \quad (2\pi)^4 m^2 |t(\mathbf{p'} \leftarrow \mathbf{p})|^2 \ , \tag{26}$$

where m is the mass of the particle.

Sometimes the entire right hand side of the equation is defined as the square of the"scattering amplitude" $|f(\mathbf{p'} \leftarrow \mathbf{p})|^2$. Clearly $f(\mathbf{p'} \leftarrow \mathbf{p})$ is directly proportional to $t(\mathbf{p'} \leftarrow \mathbf{p})$, and since there is no need to use both we will use $t(\mathbf{p'} \leftarrow \mathbf{p})$ exclusively. The key point in all this convoluted math is this: the quantity measured in an practical scattering experiment, $\frac{d\sigma}{d\Omega}$, is directly proportional to the complex square of the on shell T matrix element. The majority of Sections 4 and 5 of this paper is devoted to calculating the T matrix element, in order to check to see if an experimenter could measure a difference in cross section between like and unlike helicity collisions. Therefore, we need some tools to calculate the T operator in terms of the scattering interaction.

3.3 The T Operator

To relate the T operator to the interaction potentials, we begin by defining the Green's Operator,

$$G(z) \quad = \quad (z - H)^{-1} = (z - H^0 - V)^{-1} \ . \tag{27}$$

Here z is any (possibly complex) number such that the inverse exists. For instance, $z - H$ cannot be inverted when z is an eigenvalue of the Hamiltonian, $z = E$. Then we say that $G(z)$ has a simple pole at $z = E$, and we can avoid this pole when necessary by considering instead $z = E + i\epsilon$, and take the limit as $\epsilon \rightarrow 0$ through positive values. If we have a full description of $G(z)$, including poles (places where the Hamiltonian has discrete eigenvalues) and branch cuts (where the Hamiltonian has continuous eigenvalues), then this is equivalent to knowing the entire energy spectrum of the Hamiltonian.

It is also useful to have in hand the free Green's operator,

$$G^0 \quad = \quad (z - H^0)^{-1} \ , \tag{28}$$

and its relationship with $G(z)$,

$$G(z) \quad = \quad G^0(z) + G^0(z) V G(z) \ . \tag{29}$$

This last relation is proved via the operator identity

$$A^{-1} \quad = \quad B^{-1} + B^{-1}(B - A)A^{-1} \ . \tag{30}$$

We next define the operator $T(z)$ in terms of the Green's operator,

$$T(z) \quad = \quad V + V G(z) V \ . \tag{31}$$

It can be shown from equation (29) that

$$T(z) = V + V G^0 T(z) \ . \tag{32}$$

This last relationship is especially important, because the action of the free Green's operator on momentum states is known,

$$\langle \mathbf{p}' | G^0(z) | \mathbf{p} \rangle \quad = \quad \frac{\delta(\mathbf{p'} - \mathbf{p})}{z - p^2/2m} \ . \tag{33}$$

By inserting equation (32) into itself again and again, a series solution is formed in terms of just the potential and the free Green's operator,

$$T(z) \quad = \quad V + VG^0(z)V + VG^0(z)VG^0(z)V + \cdots \ . \tag{34}$$

This series is known as the Born series, and assuming it converges, $T(z)$ can be approximated by truncating the series at an appropriate point. Taking the first term alone is known as the first Born approximation, the first two alone are the second Born approximation, and so on.

The Born series is used frequently in scattering problems due to the difficulty or even impossibility of calculating the matrix elements of $T(z)$. It will be used here as the exclusive method for calculating cross sections and helcity asymmetries, under the assumption that all potentials selected for the study (see sections 4 and 5 below) converge sufficiently rapidly for the first term or two to dominate. For high energy processes, the Born series does serve as a good approximation [5], and since the models here are meant to evoke high energy scattering of protons, using such a power series approximation gives further parallels between the models used here and those processes.

Finally, we must prove the relationship between this $T(z)$ operator and our on shell T operator described previously. By writing out the scattering operator in terms of the Møller operators, and those in turn in terms of the unitary operators, a series of mathematical exercises and tricks can be used to show that

$$\langle \mathbf{p}'|S|\mathbf{p}\rangle \quad = \quad \delta(\mathbf{p}' - \mathbf{p}) - \lim_{\epsilon \to 0^+} 2\pi i \ \delta(E_{p'} - E_p)\langle \mathbf{p}'|T(E_p + i\epsilon)|\mathbf{p}\rangle \ . \tag{35}$$

Now it can be seen that what was called T before is in fact the seemingly arbitrarily defined $T(z)$ operator when $z = E_p + i\epsilon$. We can see that the addition of the infinitesimal positive ϵ is key, since it allows us to avoid the values of z where $T(z)$ does not exist, and allows us to use the tools of contour integration when we are confronted with integrating terms like the denominator of equation (33) over a full range of momenta.

For the remainder of this paper, it will be more convenient to write just z instead of $E_p + i\epsilon$ and just T instead of $T(E_p + i\epsilon)$.

3.4 Multichannel Scattering

In single channel scattering, the assumption was made that just a single particle was scattering off of a potential. These results can be generalized to an elastic collision between two particles by using center of mass coordinates, but that is as far single channel scattering can go. In order to investigate the scattering of bound states, we must generalize to multichannel scattering.

We take as an example a single particle scattering off of a pair of particles bound together. There are many possible outcomes of this scattering. The lone particle can scatter elastically off the pair, it can excite the bound state, and it can break the pair apart so that there are three free particles in the final state. There can even be a case where the lone particle knocks one of the members of the bound state out and takes its place. Each possible end state represents a different"channel" that the initial state can evolve into. Adding to this all of the different possible initial states that the three particles can begin in, there is a large number of channels even for this relatively simple picture.

Thankfully generalizing to the multichannel picture is straightforward. We now have a variety of $|\psi_{in}\rangle$ and $|\psi_{out}\rangle$ states to consider as asymptotes, and these asymptotes no longer have to evolve under the free Hamiltonian. Rather, the Hamiltonian of an asymptote is the full Hamiltonian with all interactions between particles that are allowed to move far apart subtracted out. For instance, in the case of a lone particle scattering off the bound state, $|\psi_{in}\rangle$ will evolve under a Hamiltonian which includes the interaction between the two bound particles, but no other potentials. In the case of an out asymptote $|\psi_{out}\rangle$ where all the particles are free (the breakup channel), the appropriate Hamiltonian is once again the free particle Hamiltonian.

Since the Hamiltonians of the asymptotes depend on the particular channel under consideration, the Møller operators depend on the in and out channels, and so also does the scattering operator. We now combine one scattering operator for each combination of in and out channel into a larger scattering operator with two indices,

$$|\psi_{out}\rangle \quad = \quad S^{\beta\alpha}|\psi_{in}\rangle \ , \tag{36}$$

where α is some label corresponding to the in asymptote and β corresponds to the out asymptote.

The matrix element of the scattering operator for a particular transition between an in channel and out channel is written as

$$\langle \underline{\mathrm{p}}', \beta|S^{\beta\alpha}|\underline{\mathrm{p}}, \alpha\rangle \quad = \quad \delta(\underline{\mathrm{p}}' - \underline{\mathrm{p}})\delta_{\beta\alpha} - 2\pi i \ \delta(E_{p'} - E_p)\langle \underline{\mathrm{p}}', \beta|T^{\beta\alpha}|\underline{\mathrm{p}}, \alpha\rangle \ , \tag{37}$$

where a convention to be used throughout this paper has been introduced, where the set of all momenta in a state is represented with a bar beneath the momentum variable. For instance, $\underline{\mathrm{k}} = \mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{P}$. The δ function between the momentum sets then corresponds to the set of all δ functions between the various momenta. We see here that just as we have a different scattering operator for each combination of in and out channels, we have a different T operator for each set of channels. It is defined as

$$T^{\beta\alpha}(z) \quad = \quad V^{\alpha} + V^{\beta}G(z)V^{\alpha} \ , \tag{38}$$

where here V^{α} refers to the sum of potentials between particle bunches in the in channel, which are also the potentials subtracted from the full Hamiltonian to get the Hamiltonian the in channel evolves under. Similarly, V^{β} refers to the sum of potentials in the out channel between particle bunches, and again these are the potentials subtracted from the full Hamiltonian to give the out channel Hamiltonian. This notation is convenient to write but can certainly be confusing. In our example of a single particle scattering off of a bound state, V^{α} is the sum of potentials with which the single particle will eventually interact with the bound state, and does not include the potential holding the bound state together. If the lone particle was to then break up the bound state, then V^{β} would include all of the potentials.

With the idea of channel Hamiltonians,

$$H^{\alpha} \quad = \quad H - V^{\alpha} \ , \tag{39}$$

we can define the channel Green's operators, which are hopefully (but not usually) more tractable than the full Green's operator appearing in equation (38). They are

$$G^{\alpha}(z) \quad = \quad (z - H^{\alpha})^{-1} \ . \tag{40}$$

By using the same operator identity as in the single channel case equation (30), we can write an equation for $T^{\beta\alpha}(z)$ in terms of the out channel Green's operator,

$$T^{\beta\alpha}(z) \quad = \quad V^{\alpha} + V^{\beta} G^{\beta}(z) T^{\beta\alpha} \ . \tag{41}$$

This relationship can be inserted into itself again and again just as before, generating an infinite series for the multichannel scattering process analogous to the Born series for the single channel process. This series is

$$T^{\beta\alpha}(z) \quad = \quad V^{\alpha} + V^{\beta} G^{\beta}(z) V^{\alpha} + V^{\beta} G^{\beta}(z) V^{\beta} G^{\beta}(z) V^{\alpha} + \cdots \ . \tag{42}$$

The breakup channel is especially convenient when using this series, since in that case G^{β} is just the free Green's operator, as discussed above.

The final consideration in adapting single channel scattering to multichannel scattering is the calculation of the differential cross section. Unfortunately there is no clean generalization. The derivation of each differential cross section depends on the particulars of out channel, an also the particular question asked. For the example of the lone particle and bound state collision, one might ask for the cross section for finding one of the particles in a differential unit of solid angle without caring where any of the other particles are found. Alternately, one might ask where one of the particles is found given that the other two are detected with specific energies in specific places. This second example of differential cross section is the most specific question that can be asked. It is, in a sense, the "most differential" of all the cross sections, and all others can be built up from it by integrating over the variables that the experimenter is uninterested in or unable to measure. One result that carries through all scattering processes and channels choices is that this "most differential" cross section (denoted here as $\frac{D\sigma}{D\Omega}$) is proportional to the square of the T operator bracketed between the initial and final states,

$$\frac{D\sigma}{D\Omega} \quad \propto \quad |\langle final|T^{\beta\alpha}|initial\rangle|^2 \ . \tag{43}$$

The objective of this paper is to compare the differential cross sections between like and unlike helicity collisions, so the constants of proportionality are unimportant. They depend on the particular case under consideration, and will divide out in the final asymmetry calculation, so they will not always be given.

4 The Three Body Problem

The first step in investigating helicity effects in quantum scattering is to see whether such effects are present in a simple and easily calculated model. It also serves as a warmup for the four body scattering problem which is the main topic of this thesis.

4.1 The Model

We imagine that we have two particles, particles 1 and 2, both with mass m. Initially, particle 2 is bound to a third particle whose mass M is much greater than m, so that (in the limit that M is infinite) particle 3 sits at the origin. The initial bound state between particles 2 and 3 is denoted ψ, and the interaction between 2 and 3 is assumed to be spherically symmetric. Particle 1 is initially a plane wave, and interacts with particle 2 via a Yukawa potential V_{12}. It is assumed that the coupling constant is small, so that the Born series converges rapidly, and the first term serves as a good approximation.

Note that since particle 3 remains at the origin, this model is identical to a two body problem with particle 2 bound to the origin. We do not enforce conservation of momentum in this problem, because particle 3 can freely absorb any unaccounted for momentum.

We are interested in the breakup channel, where in the final state both particles 1 and 2 are plane waves. Their final momenta will be $\mathbf{k_1}$ and $\mathbf{k_2}$, respectively, and particle 1 will have an initial momentum $\mathbf{k_{0,1}}$

4.2 The First Born Approximation

The first Born approximation for this breakup process is the first term in the infinite series of equation (42), and is denoted by T_1. It's matrix element is given by

$$\begin{aligned} \langle \underline{\mathrm{k}} | T_1 | i \rangle &= \langle \mathbf{k_1}, \mathbf{k_2} | V_{12} | \mathbf{k_{0,1}}, \psi \rangle & (44) \\ &= \int d\mathbf{x_1}\, d\mathbf{x_2}\, d\mathbf{x'_1}\, d\mathbf{x'_2} \langle \mathbf{k_1}, \mathbf{k_2} | \mathbf{x_1}, \mathbf{x_2} \rangle \langle \mathbf{x_1}, \mathbf{x_2} | V_{12} | \mathbf{x'_1} \mathbf{x'_2} \rangle \langle \mathbf{x_1}, \mathbf{x_2} | \mathbf{p_1} \psi \rangle & (45) \\ &= (2\pi\hbar)^{-9/2} \int d\mathbf{x_1}\, d\mathbf{x_2}\, e^{-i(\mathbf{k_1}\cdot\mathbf{x_1} + \mathbf{k_2}\cdot\mathbf{x_2} - \mathbf{k_{0,1}}\cdot\mathbf{x_1})/\hbar}\, V_{12}(\mathbf{x_2} - \mathbf{x_1}) \psi(\mathbf{x_2}) \ . & (46) \end{aligned}$$

To solve this, we make the shift substitution $\mathbf{x} = (\mathbf{x_2} - \mathbf{x_1})$, $\mathbf{X} = \mathbf{x_2}$, rearrange the terms, and use the Fourier transform of the Yukawa potential, which gives

$$\begin{aligned} &= (2\pi\hbar)^{-9/2} \int d\mathbf{x} d\mathbf{X}\ e^{-i(\mathbf{k_1} - \mathbf{k_{01}})\cdot\mathbf{x}/\hbar}\ e^{-i(\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}})\cdot\mathbf{X}/\hbar}\, V(\mathbf{x}) \psi(\mathbf{X}) & (47) \\ &= (2\pi\hbar)^{-3} \frac{\sqrt{2\hbar/\pi}}{\beta^2\hbar^2 + |\mathbf{k_1} - \mathbf{k_{0,1}}|^2} \int d\mathbf{X}\ e^{-i(\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}})\cdot\mathbf{X}/\hbar}\ r_{E,l}(X)\, Y_{l,m}(\hat{\mathbf{X}}) \ , & (48) \end{aligned}$$

where the hat above the vector $\mathbf{X}$ indicates the direction of $\mathbf{X}$, a shorthand used because the spherical harmonic is a function only of the angles that give the orientation of $\mathbf{X}$, and not the magnitude of the vector. A larger hat is used over a combination of vectors to indicate that the spherical harmonic is a function of the angles that orient the resulting vector, such as $Y_{l,m}(\widehat{\mathbf{p_1} + \mathbf{p_2}})$. Also, in the above expression we have separated out the angular and radial parts of the wavefunction,

$$\psi(\mathbf{x}) \quad = \quad r_{E,l}(X) Y_{l,m}(\hat{\mathbf{X}}) \ , \tag{49}$$

because the potential binding particle 2 to the origin is spherically symmetric. Note $r_{E,l}(X)$ is left unspecified, and depends on the particular potential about the origin.

To finish the integration, we make use of the Bessel function expansion of the exponential,

$$\begin{aligned} e^{-i(\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}})\cdot\mathbf{X})/\hbar} &= 4\pi \sum_{l',m'} i^{l'}\ j_{l'}(|\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}|\ X/\hbar)\ Y_{l',m'}(\widehat{\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}}) \\ &\qquad \times Y^*_{l',m'}(\hat{X}) \ . & (50) \end{aligned}$$

Inserting this expansion into equation (48) and integrating over the angles of $\mathbf{X}$ eliminates the spherical harmonics, producing a Kronecker δ, giving

$$\begin{aligned}
\langle \underline{\mathrm{k}} | T^{\beta\alpha} | i \rangle &= (2\pi\hbar)^{-3/2} \int X^2 \, dX \frac{2/(\hbar\pi)}{\beta^2\hbar^2 + |\mathbf{k_1} - \mathbf{k_{0,1}}|^2} \sum_{l',m'} i^{l'} \; j_{l'}(|\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}| \; X/\hbar) \\
&\quad \times Y_{l',m'}(\widehat{\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}}) \; \delta_{l',l}\delta_{m',m} \; r_{E,l}(X) \qquad (51) \\
&= \frac{2/(\hbar\pi)}{\beta^2\hbar^2 + |\mathbf{k_1} - \mathbf{k_{0,1}}|^2} \; i^l \; Y_{l,m}(\widehat{\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}}) R_{E,l}(|\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}|) \; , \qquad (52)
\end{aligned}$$

where

$$R_{E,l}(|\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}|) = (2\pi\hbar)^{-3/2} \int dX \; X^2 \; j_l(|\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}| \; X/\hbar) r_{E,l}(X) \; . \qquad (53)$$

4.3 Helicity Dependence in the First Born Approximation

The cross section in the first Born is given by

$$\begin{aligned}
\frac{d^3\sigma}{d\Omega_1 d\Omega_2 dE_1} &= \frac{16\pi^4 m^3 k_1 k_2}{k_{1,0}} |\langle \underline{\mathrm{k}} | T_1 | i \rangle|^2 \qquad (54) \\
&= \frac{64\pi^2 m^3 k_1 k_2/(k_{1,0}\hbar^2)}{(\beta^2\hbar^2 + |\mathbf{k_1} - \mathbf{k_{0,1}}|^2)^2} \; |Y_{l,m}(\widehat{\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}})|^2 R_{E,l}(|\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}|)^2 \; . \qquad (55)
\end{aligned}$$

Note $|Y_{l,m}((\widehat{\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_{0,1}}}))|^2$ does not depend on the sign of m. In this case, we take the asymmetry between a positive value of m (particle 2 circulates in one direction) and a negative value of m (particle 2 circulates the other way). From this we immediately find that

$$ASM = \frac{\frac{d^3\sigma}{d\Omega_1 d\Omega_2 dE_1}_+ - \frac{d^3\sigma}{d\Omega_1 d\Omega_2 dE_1}_-}{\frac{d^3\sigma}{d\Omega_1 d\Omega_2 dE_1}_+ + \frac{d^3\sigma}{d\Omega_1 d\Omega_2 dE_1}_-} = 0 \; . \qquad (56)$$

There is no helicity asymmetry in this interaction, at least to the first approximation.

5 The Four Body Problem

Rather than continue to calculate the three body problem to greater orders of approximation, the focus of the rest of the paper will be on investigating asymmetry in a problem analogous to the scattering of polarized protons. This situation is that of scattering two bound states, each containing angular momentum. The two bound states will be identical to each other, except that one may have the opposite helicity of the other.

The four body problem mirrors the semi-classical problems already investigated for asymmetries, and at first glance it seems that an asymmetry is expected. In order to begin the search, we first specify the exact scattering model we will use, and the motivations for it.

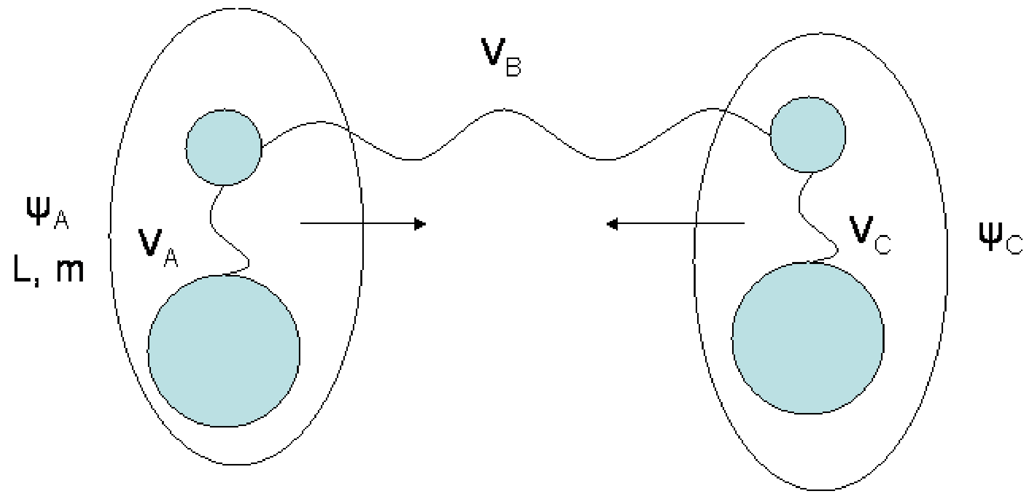

Figure 2: The model for the four body problem. The two bound states ψ_A and ψ_C have some initial angular momentum, and V_A and V_C are the potentials binding each of the two states together. V_B acts between particles 1 and 3.

5.1 The Model

There are two important considerations in choosing a scattering model for the helicity study. The first is to choose a model with enough of an analogy to the process of polarized proton-proton collisions that any results can be taken to give reasonable qualitative predictions about that process. The second consideration is to choose a model for which the cross section can be calculated with a reasonable amount of effort.

For this model each bound state consists of a lighter particle bound to a heavier particle. The lighter particles will interact within their bound state and with each other, while the heavier particles will not interact between bound states. This picture is meant to evoke the scattering between two partons, each with a small fraction of the overall momentum of the proton, and each bound to a larger body of noninteracting partons. The reasoning behind this picture is that in high energy proton-proton collisions, two of the partons can interact strongly, transferring a large amount of momentum. When this occurs the two partons scatter at large angles, while the rest of each the proton continues traveling in its original direction of motion, initially unaffected by the interaction. Reference [6] gives an example discussion of the case of electron-proton collisions.

The less massive particle in each pair has mass m, and the larger mass is M. There are three potentials to consider. V_A binds the first pair of particles, particles 1 and 2. V_C acts between the second pair of particles, particles 3 and 4. 1 and 3 are the lighter particles, and they will interact via a potential V_B which will completely ignore particles 2 and 4. See Figure 2 for a schematic of the interaction.

5.2 The Coordinate System

First we must define the relative coordinates for this system. They are

$$\mathbf{r_1} = \mathbf{x_2} - \mathbf{x_1} \ , \tag{57}$$

$$\mathbf{r_2} = \mathbf{x_4} - \mathbf{x_3} \ , \tag{58}$$

$$\mathbf{R} = \frac{m\mathbf{x_3} + M\mathbf{x_4} - m\mathbf{x_1} - M\mathbf{x_2}}{m+M} , \quad (59)$$

$$\mathbf{X} = \frac{m\mathbf{x_1} + M\mathbf{x_2} + m\mathbf{x_3} + M\mathbf{x_4}}{2(m+M)} . \quad (60)$$

The conjugate momenta are

$$\mathbf{k_1} = \frac{m\mathbf{p_2} - M\mathbf{p_1}}{m+M} , \quad (61)$$

$$\mathbf{k_2} = \frac{m\mathbf{p_4} - M\mathbf{p_3}}{m+M} , \quad (62)$$

$$\mathbf{K} = (\mathbf{p_3} + \mathbf{p_4} - \mathbf{p_1} - \mathbf{p_2})/2 , \quad (63)$$

$$\mathbf{P} = \mathbf{p_1} + \mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4} . \quad (64)$$

The potentials V_A and V_C act on the relative position coordinates of $\mathbf{x_2} - \mathbf{x_1} = \mathbf{r_1}$ and $\mathbf{x_4} - \mathbf{x_3} = \mathbf{r_2}$, respectively. They have a natural representation in the set of momenta coordinates,

$$\langle \mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{P} | V_A | \mathbf{k_1'}, \mathbf{k_2'}, \mathbf{K'}, \mathbf{P'} \rangle = V_A(\mathbf{k_1}, \mathbf{k_1'}) \delta(\mathbf{k_2} - \mathbf{k_2'}) \delta(\mathbf{K} - \mathbf{K'}) \delta(\mathbf{P} - \mathbf{P'}) , \quad (65)$$

$$\langle \mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{P} | V_C | \mathbf{k_1'}, \mathbf{k_2'}, \mathbf{K'}, \mathbf{P'} \rangle = V_C(\mathbf{k_2}, \mathbf{k_2'}) \delta(\mathbf{k_1} - \mathbf{k_1'}) \delta(\mathbf{K} - \mathbf{K'}) \delta(\mathbf{P} - \mathbf{P'}) . \quad (66)$$

The potentials act only on the coordinate kets that they depend on, and ignore the other coordinates, so that those kets form delta functions. Unfortunately the potential V_B depends on the coordinate $\mathbf{x_3} - \mathbf{x_1} = \mathbf{R} + \frac{M}{M+m}(\mathbf{r_1} - \mathbf{r_2})$, and so it has a more complicated representation in this set of momenta. The representation is

$$\begin{aligned} \langle \mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{P} | V_B | \mathbf{k_1'}, \mathbf{k_2'}, \mathbf{K'}, \mathbf{P'} \rangle &= V_B(\frac{m}{m+M}\mathbf{K} - (\mathbf{k_2} - \mathbf{k_1})/2, \frac{m}{m+M}\mathbf{K'} - (\mathbf{k_2'} - \mathbf{k_1'})/2) \\ &\times \delta(\frac{M}{m+M}\mathbf{K} + (\mathbf{k_2} - \mathbf{k_1})/2 - \frac{M}{m+M}\mathbf{K'} - (\mathbf{k_2'} - \mathbf{k_1'})/2) \\ &\times \delta(\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_1'} - \mathbf{k_2'}) \delta(\mathbf{P} - \mathbf{P'}) . \quad (67) \end{aligned}$$

A complete derivation of this relation is given in Appendix A.

5.3 The Separable Potentials

In order to ease the difficulty of calculating the four body scattering asymmetry, mathematically convenient potentials known as separable potentials will be used. Separable potentials have the form

$$\langle r | V | r' \rangle = -\frac{\lambda}{2\mu} \langle r | g \rangle \langle g | r' \rangle \quad (68)$$

$$= -\frac{\lambda}{2\mu} g(r) g^*(r') \quad (69)$$

where μ represents the reduced mass of the two body interaction. The potential in the position representation does not contain a delta function $\delta(r - r')$ and so is a nonlocal potential. A nonlocal potential can act instantaneously, and so violates the tenants of relativity. However, as demonstrated by Yamaguchi [7], potentials of this form can be used to solve many scattering problems exactly, and for the purpose of this paper will serve to make the equations tractable. Yamaguchi demonstrated that the solutions given by this type of potential could indeed match nuclear scattering data very well, and so they serve as a helpful tool for the study of quantum scattering.

The particular type of nonlocal, separable potentials studied here are the Yamaguchi form factors. The following builds upon the equations and formalism presented in [7] and in [8]. The form factors are

$$\langle r|g_l\rangle = \left(\frac{i}{2}\right)^l \frac{r^{l-1}}{l!}\, e^{-\beta r}\ , \tag{70}$$

where the case $l = 0$ gives the Yukawa potential. For our purposes we must include angular momentum in the bound states, and so we consider these functions multiplied by the spherical harmonics,

$$\langle \mathbf{r}|g_l\rangle = \left(\frac{i}{2}\right)^l \frac{r^{l-1}}{l!}\, e^{-\beta r}\ Y_{l,m}(\hat{\mathbf{r}})\ . \tag{71}$$

We will use the momentum representation of these for factors exclusively, and so we need to Fourier transform them, again using the Bessel function expansion of the exponential,

$$\begin{aligned}
\langle \mathbf{p}|g_l\rangle &= \int d\mathbf{r}\langle \mathbf{p}|\mathbf{r}\rangle\langle \mathbf{r}|g_l\rangle & (72)\\
&= (2\pi\hbar)^{-3/2}\int d\mathbf{r}\left(\frac{i}{2}\right)^l \frac{r^{l-1}}{l!}\ e^{-\beta r}\, Y_{l,m}(\hat{\mathbf{r}})e^{-i\mathbf{p}\cdot\mathbf{r}} & (73)\\
&= \sqrt{\frac{2}{\pi}}\hbar^{-3/2}\ (1/2)^l Y_{l,m}(\hat{\mathbf{p}})\int dr\frac{r^{l+1}}{l!}e^{-\beta r}j_l(pr/\hbar) & (74)\\
&= \hbar^{-3/2}\ (1/2)^l Y_{l,m}(\hat{\mathbf{p}})\sqrt{\frac{\hbar}{p}}\int dr\frac{r^{l+1/2}}{l!}e^{-\beta r}J_{l+1/2}(pr/\hbar)\ . & (75)
\end{aligned}$$

The integral can be evaluated [9], using

$$\int_0^\infty dr e^{-\beta r}J_\nu(\alpha r)r^\nu = \frac{(2\alpha)^\nu\Gamma(\nu+1/2)}{\sqrt{\pi}(\beta^2+\alpha^2)^{\nu+1/2}}\ . \tag{76}$$

Inserting this and evaluating, we finally have

$$\langle \mathbf{p}|g_l\rangle = \sqrt{\frac{2\hbar}{\pi}}\frac{\hbar^l p^l Y_{l,m}(\hat{\mathbf{p}})}{(\hbar^2\beta^2+p^2)^{l+1}}\ . \tag{77}$$

The first case to consider is the potential V_B. This is the interaction between particles 1 and 3, for which we will use the simple case where $l = 0$, which just reduces to the Yukawa potential. It is given as

$$
\begin{aligned}
\langle \mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{P}| V_B |\mathbf{k_1'}, \mathbf{k_2'}, \mathbf{K'}, \mathbf{P'}\rangle &= -\frac{\lambda_B}{m}\ \delta(\mathbf{P}-\mathbf{P'})\delta(\mathbf{k_1}+\mathbf{k_2}-\mathbf{k_1'}-\mathbf{k_2'}) \\
&\quad \times\delta(\frac{M}{m+M}\mathbf{K}+(\mathbf{k_2}-\mathbf{k_1})/2-\frac{M}{m+M}\mathbf{K'}-(\mathbf{k_2'}-\mathbf{k_1'})/2) \\
&\quad \times B(\frac{m}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2) \\
&\quad \times B^*(\frac{m}{m+M}\mathbf{K'}-(\mathbf{k_2'}-\mathbf{k_1'})/2)\ , \qquad (78) \\
B(\frac{m}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2) &= \frac{\sqrt{\frac{\hbar/2}{\pi}}}{\gamma^2\hbar^2+|\frac{m}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2|^2}\ , \qquad (79)
\end{aligned}
$$

where γ has been used for the constant determining the characteristic range of V_B.

The next step is to consider a potential that does contain angular momentum, and so we look at the potential V_A. Its matrix element is

$$
\begin{aligned}
\langle \mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{P}| V_A |\mathbf{k_1'}, \mathbf{k_2'}, \mathbf{K'}, \mathbf{P'}\rangle &= -\frac{\lambda_A}{2\mu}\ \delta(\mathbf{P}-\mathbf{P'})\delta(\mathbf{K}-\mathbf{K'})\delta(\mathbf{k_2}-\mathbf{k_2'}) \\
&\quad \times\sum_m A_m(\mathbf{k_1})A_m^*(\mathbf{k_1'})\ , \qquad (80) \\
A_m(\mathbf{k_1}) &= \sqrt{\frac{2\hbar}{\pi}}\ \hbar^l k_1^l(k_1^2+\hbar^2\beta^2)^{-l-1}Y_{l,m}(\widehat{\mathbf{k_1}})\ , \qquad (81) \\
\mu &= \frac{Mm}{m+M}\ . \qquad (82)
\end{aligned}
$$

Now V_C is identical to V_A save that it acts only within its pair of particles, on coordinate $\mathbf{k_2}$. So, its matrix element is

$$
\begin{aligned}
\langle \mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{P}| V_C |\mathbf{k_1'}, \mathbf{k_2'}, \mathbf{K'}, \mathbf{P'}\rangle &= -\frac{\lambda_C}{2\mu}\ \delta(\mathbf{P}-\mathbf{P'})\delta(\mathbf{K}-\mathbf{K'})\delta(\mathbf{k_1}-\mathbf{k_1'}) \\
&\quad \times\sum_m C_m(\mathbf{k_2})C_m^*(\mathbf{k_2'})\ , \qquad (83) \\
C_m(\mathbf{k_2}) &= \sqrt{\frac{2\hbar}{\pi}}\ \hbar^l k_2^l(k_2^2+\hbar^2\beta^2)^{-l-1}Y_{l,m}(\widehat{\mathbf{k_2}})\ . \qquad (84)
\end{aligned}
$$

It is important to note here that usually a separable potential supports only a single bound state. The sum over different m quantum numbers here introduces a degeneracy which, as will be shown below, allows us to have bound states with varying m quantum numbers arise from the same interaction. This is crucial to studying collisions where the bound states have differing m. The A_m and C_m functions do not have coefficients in front of them simply because when they have equal weight, the potentials are in a sense as spherically symmetric as they can be. There is no such summation over different l quantum numbers, which means that there is only a single possible l state for each separable potential. Since the investigation focuses on like and unlike helicity, determined by the m quantum numbers, there is no issue with the inability to have bound states with differing l arise from the same interaction.

With these in hand, we can calculate the bound states, beginning with $\langle \mathbf{k_1}|\psi_A\rangle$. First we write the Schröedinger Equation in the momentum representation, for a bound state,

$$\langle \mathbf{k_1}|\psi_A\rangle \quad = \quad \psi_A(\mathbf{k_1}) \; . \tag{85}$$

The equation, with the bound state energy equal to $-\alpha^2/2\mu$ (the minus sign is required for a bound state solution), is

$$\frac{k_1^2}{2\mu}\psi_A(\mathbf{k_1}) + \int d\mathbf{k_1'} \langle \mathbf{k_1}|\, V_A|\mathbf{k_1'}\rangle\psi_A(\mathbf{k_1'}) \quad = \quad -\frac{\alpha^2}{2\mu}\psi_A(\mathbf{k_1}) \; . \tag{86}$$

Rearranging and inserting our definition for the potential, the equation becomes

$$\begin{aligned}(k_1^2+\alpha^2)\psi_A(\mathbf{k_1}) \quad &= \quad \lambda_A \, \frac{2\hbar^{(2l+1)}}{\pi}\frac{k_1^l}{(k_1^2+\hbar^2\beta^2)^{l+1}} \, \sum_m Y_{l,m}(\widehat{\mathbf{k_1}}) \\ &\quad \times \int d\mathbf{k_1'}\frac{k_1'^l}{(k_1'^2+\hbar^2\beta^2)^{l+1}}\, Y^*_{l,m}(\widehat{\mathbf{k_1'}})\psi_A(\mathbf{k_1'}) \; . \end{aligned} \tag{87}$$

The solution is

$$\psi_A(\mathbf{k_1}) \quad = \quad N\sum_m a_m \frac{k_1^l \; Y_{l,m}(\widehat{\mathbf{k_1}})}{(k_1^2+\hbar^2\beta^2)^{l+1}(k_1^2+\alpha^2)} \; , \tag{88}$$

where N is the normalization factor, given by

$$\frac{1}{N^2} \quad = \quad \int_0^\infty dp\frac{p^{2(l+1)}}{(p^2+\hbar^2\beta^2)^{2(l+1)}(p^2+\alpha^2)^2} \; , \tag{89}$$

and the squares of the coefficients a_m sum to one,

$$\sum_m |a_m|^2 \quad = \quad 1 \; . \tag{90}$$

Inserting this solution into equation (87) both confirms that it is a solution and gives an expression relating λ_A to α. Inserting, and integrating over the spherical harmonics to eliminate one summation, we have

$$\begin{aligned} N\sum_m a_m \frac{k_1^l \; Y_{l,m}(\widehat{\mathbf{k_1}})}{(k_1^2+\hbar^2\beta^2)^{l+1}} \quad &= \quad N \; \lambda_A \; \frac{2}{\pi}\frac{k_1^l}{(k_1^2+\hbar^2\beta^2)^{l+1}} \\ &\quad \times \sum_m a_m \; Y_{l,m}(\widehat{\mathbf{k_1}}) \int_0^\infty dk_1 \frac{k_1^{2(l+1)}}{(k_1^2+\hbar^2\beta^2)^{2(l+1)}(k_1^2+\alpha^2)} \; . \end{aligned} \tag{91}$$

Cancelling from the right and the left sides, and rewriting the variable of integration, we see that the condition that the equality holds is that

$$\frac{\pi}{2\lambda_A} = \int_0^\infty dp \frac{p^{2(l+1)}}{(p^2+\hbar^2\beta^2)^{2(l+1)}(p^2+\alpha^2)} . \tag{92}$$

This defines λ_A.

The relations for V_C and it's initial state hold in exactly the same way, with A being replaced by C and $\mathbf{k_1}$ by $\mathbf{k_2}$. Since the bound states are identical but for a possible difference in the superposition of states with differing m quantum numbers, the binding energies are equal and thus $\lambda_A = \lambda_C$.

The formula for $\langle \mathbf{k_2}|\psi_C\rangle = \psi_C(\mathbf{k_2})$ is then

$$\psi_C(\mathbf{k_2}) = N \sum_m a_m \frac{k_2^l \; Y_{l,m}(\widehat{\mathbf{k_2}})}{(k_2^2+\hbar^2\beta^2)^{l+1}(k_2^2+\alpha^2)} . \tag{93}$$

For the purpose of the discussion of helicity, we will imagine the m value of both ψ_A and ψ_C are known, and will be equal to m_A and m_C, respectively. For like helicity collisions, we will use $m_A = m_C$, and for unlike helicity collisions $m_A = -m_C$. However for generality these substitutions will be delayed until the final calculations of helicity asymmetry. With this choice,

$$\psi_A(\mathbf{k_1}) = N \frac{k_1^l \; Y_{l,m_A}(\widehat{\mathbf{k_1}})}{(k_1^2+\hbar^2\beta^2)^{l+1}(k_1^2+\alpha^2)} \equiv R_l(k_1)\; Y_{l,m_A}(\hat{\mathbf{k_1}}) , \tag{94}$$

$$\psi_C(\mathbf{k_2}) = N \frac{k_2^l \; Y_{l,m_C}(\widehat{\mathbf{k_2}})}{(k_2^2+\hbar^2\beta^2)^{l+1}(k_2^2+\alpha^2)} \equiv R_l(k_2)\; Y_{l,m_C}(\hat{\mathbf{k_2}}) , \tag{95}$$

where the functions R_A and R_C are the "radial" portions of the wavefunction, those parts which depend only on the magnitude of the momenta. This abbreviation makes the m dependence of each of the initial state wavefunctions explicit.

5.4 The First Born Approximation

With the preliminaries in hand, we again use the first Born approximation to calculate the differential cross section of the breakup interaction, and so determine the effects of helicity on the interaction. Here, the series is essentially a power series in terms of the coupling constants divided by the characteristic masses (either the masses of the particles or the appropriate reduced mass). The size of this ratio determines how fast the series converges and how good the first terms are as an approximation. Here we will let $\frac{\lambda_i}{m} << 1$ so that the series approximation is a good one.

For the breakup interaction, the matrix element at first order in the series is

$$\langle \underline{\mathrm{k}}|T_1|i\rangle = \langle \underline{\mathrm{k}}|V_B|i\rangle . \tag{96}$$

Inserting complete sets of states as appropriate, the T matrix element is

$$\begin{aligned} \langle \underline{\mathrm{k}}|V_B|i\rangle &= \int d\underline{\mathrm{k}}' \langle \underline{\mathrm{k}}|V_B|\underline{\mathrm{k}}'\rangle\langle k'|i\rangle \qquad (97) \\ &= -\frac{\lambda_B}{m}\; B(\frac{m}{m+M}\mathbf{K} - (\mathbf{k_2}-\mathbf{k_1})/2) \int d\mathbf{k_1'}\, d\mathbf{k_2'}\, d\mathbf{K}'\, B^*(\frac{m}{m+M}\mathbf{K}' - (\mathbf{k_2'}-\mathbf{k_1'})/2) \end{aligned}$$

$$\times\delta(\mathbf{k_1}+\mathbf{k_2}-\mathbf{k_1'}-\mathbf{k_2'})\delta(\frac{M}{m+M}\mathbf{K}+(\mathbf{k_2}-\mathbf{k_1})/2-\frac{M}{m+M}\mathbf{K'}-(\mathbf{k_2'}-\mathbf{k_1'})/2)$$
$$\times\langle\mathbf{k_1'},\mathbf{k_2'},\mathbf{K'},\mathbf{P}|i\rangle \ . \tag{98}$$

Inserting the definition for the initial state,

$$\begin{aligned}\langle\underline{\mathrm{k}}|V_B|i\rangle &= -\frac{\lambda_B}{m}\,B(\frac{m}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2)\int d\mathbf{k_1'}\mathbf{dk_2'}\mathbf{dK'}B^*(\frac{m}{m+M}\mathbf{K'}-(\mathbf{k_2'}-\mathbf{k_1'})/2)\\ &\times\delta(\mathbf{k_1}+\mathbf{k_2}-\mathbf{k_1'}-\mathbf{k_2'})\delta(\frac{M}{m+M}\mathbf{K}+(\mathbf{k_2}-\mathbf{k_1})/2-\frac{M}{m+M}\mathbf{K'}-(\mathbf{k_2'}-\mathbf{k_1'})/2)\\ &\times\delta(\mathbf{K'}-\mathbf{K_0})\delta(\mathbf{P}-\mathbf{P_0})\psi_A(\mathbf{k_1'})\psi_C(\mathbf{k_2'}) \ . \end{aligned}\tag{99}$$

Resolving the delta functions in the primed variables by integrating over $\mathbf{k_1'}$ and $\mathbf{k_2'}$ we have

$$\begin{aligned}\langle\underline{\mathrm{k}}|V_B|i\rangle &= -\frac{\lambda_B}{m}\,B(\frac{m}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2)\delta(\mathbf{P}-\mathbf{P_0})\\ &\times\int d\mathbf{K'}B^*(\mathbf{K'}-\frac{M}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2)\delta(\mathbf{K'}-\mathbf{K_0})\\ &\times\psi_A(\mathbf{k_1}+\frac{M}{m+M}(\mathbf{K'}-\mathbf{K}))\ \psi_C(\mathbf{k_2}-\frac{M}{m+M}(\mathbf{K'}-\mathbf{K})) \end{aligned}\tag{100}$$

$$\begin{aligned}&= -\frac{\lambda_B}{m}\,\delta(\mathbf{P}-\mathbf{P_0})B(\frac{m}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2)B^*(\mathbf{K_0}-\frac{M}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2)\\ &\times\psi_A(\mathbf{k_1}+\frac{M}{m+M}(\mathbf{K_0}-\mathbf{K}))\ \psi_C(\mathbf{k_2}-\frac{M}{m+M}(\mathbf{K_0}-\mathbf{K})) \ . \end{aligned}\tag{101}$$

We also recognize at this point that the collision is occurring in the lab frame, so that $\mathbf{P_0}=0$ and the delta function enforces $\mathbf{P}=0$. There is no dependence on the total momentum in any of the pieces, and so we can simply drop the delta function from further calculations. With this in hand, we can calculate the helicity asymmetry in the First Born Approximation.

5.5 Helicity Asymmetry in the First Born Approximation

For this breakup problem, the most specific cross section that can be considered would be the probability of finding one of the particles in a small differential unit of solid angle given the measured positions and energies of the other three particles. This is our "most differential" cross section, and in the case of taking say particle 1 as our particle of interest is it written as

$$\frac{d^7\sigma}{d\Omega_1 d\Omega_2 dE_2 d\Omega_3 dE_3 d\Omega_4 dE_4} \equiv \frac{D\sigma}{D\Omega}\propto|\langle\underline{\mathrm{k}}|T_1|i\rangle|^2 \ . \tag{102}$$

We can immediately see that because of the square in this relationship, there is no helicity asymmetry in the first Born approximation of the four body problem, because all of the dependence on m_A and m_C is contained within the spherical harmonics, which are complex squared along with T, and the square of the spherical harmonics does not depend on the sign of the m quantum number.

More specifically, the asymmetry calculation for like ($m_A=m_C$) versus unlike ($m_A=-m_C$) helicity collisions is

$$
\begin{aligned}
ASM &= \frac{\frac{D\sigma}{D\Omega}_{like} - \frac{D\sigma}{D\Omega}_{unlike}}{\frac{D\sigma}{D\Omega}_{like} + \frac{D\sigma}{D\Omega}_{unlike}} && (103)\\
&= \frac{|\langle \underline{\mathrm{k}}|T_1|i\rangle|^2_{like} - |\langle \underline{\mathrm{k}}|T_1|i\rangle|^2_{unlike}}{|\langle \underline{\mathrm{k}}|T_1|i\rangle|^2_{like} + |\langle \underline{\mathrm{k}}|T_1|i\rangle|^2_{unlike}} && (104)\\
&= \frac{|\psi_A(\mathbf{k_1})\psi_C(\mathbf{k_2})|^2_{like} - |\psi_A(\mathbf{k_1})\psi_C(\mathbf{k_2})|^2_{unlike}}{|\psi_A(\mathbf{k_1})\psi_C(\mathbf{k_2})|^2_{like} + |\psi_A(\mathbf{k_1})\psi_C(\mathbf{k_2})|^2_{unlike}} && (105)\\
&= \frac{|P_{l,|m_A|}(cos\theta_{k_1})P_{l,|m_A|}(cos\theta_{k_2})|^2 - |P_{l,|m_A|}(cos\theta_{k_1})P_{l,|-m_A|}(cos\theta_{k_2})|^2}{|P_{l,|m_A|}(cos\theta_{k_1})P_{l,|m_A|}(cos\theta_{k_2})|^2 + |P_{l,|m_A|}(cos\theta_{k_1})P_{l,|-m_A|}(cos\theta_{k_2})|^2} && (106)\\
&= 0\ . && (107)
\end{aligned}
$$

Note that in the preceding calculation, the following were used

$$
\begin{aligned}
Y_{l,m}(\theta,\phi) &= (-1)^{(m+|m|)/2} P_{l,m}(cos\theta)e^{im\phi}\ , && (108)\\
|Y_{l,m}(\theta,\phi)|^2 &= |P_{l,m}(cos\theta)|^2\ . && (109)
\end{aligned}
$$

Clearly to first order there is no difference in the measurable quantities between like and unlike helicity collisions. However, the first Born term has ignored the effects of two of the three potentials on the collision. It seems reasonable that to answer the question of helicity asymmetry for the four body system, we need to move on to the next level of approximation. As such, we calculate the second Born approximation.

5.6 The Second Born Approximation

For the second Born approximation, we must include the first and second terms, T_1 and T_2 of the expansion (42). We have already calculated the matrix element of T_1. The matrix element of T_2 is given for this interaction by

$$
\langle \underline{\mathrm{k}}|T_2|i\rangle = \langle \underline{\mathrm{k}}|V_A\ G^0\ V_B|i\rangle + \langle \underline{\mathrm{k}}|V_B\ G^0\ V_B|i\rangle + \langle \underline{\mathrm{k}}|V_C\ G^0\ V_B|i\rangle\ . \qquad (110)
$$

Here again G^0 is the free Green's operator, and it can be shown that

$$
\langle \mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{P}|G^0|\mathbf{k_1'}, \mathbf{k_2'}, \mathbf{K'}, \mathbf{P'}\rangle = \frac{\delta(\mathbf{k_1}-\mathbf{k_1'})\ \delta(\mathbf{k_2}-\mathbf{k_2'})\ \delta(\mathbf{K}-\mathbf{K'})\ \delta(\mathbf{P}-\mathbf{P'})}{z - T(\underline{\mathrm{k}})}\ . \qquad (111)
$$

Here in a confusing but standard notation the $T(\underline{\mathrm{k}})$ in the denominator represents the kinetic energy of the four particles. It is not related to the T operator. It is

$$
T(\mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{P}) = \frac{k_1^2}{2\mu} + \frac{k_2^2}{2\mu} + \frac{K^2}{m+M} + \frac{P^2}{2(m+M)}\ . \qquad (112)
$$

Before plunging into the lengthy calculation of this second term, it is useful to get a picture in mind of what is actually being calculated. T_1 and each of the three terms in T_2 can be thought of

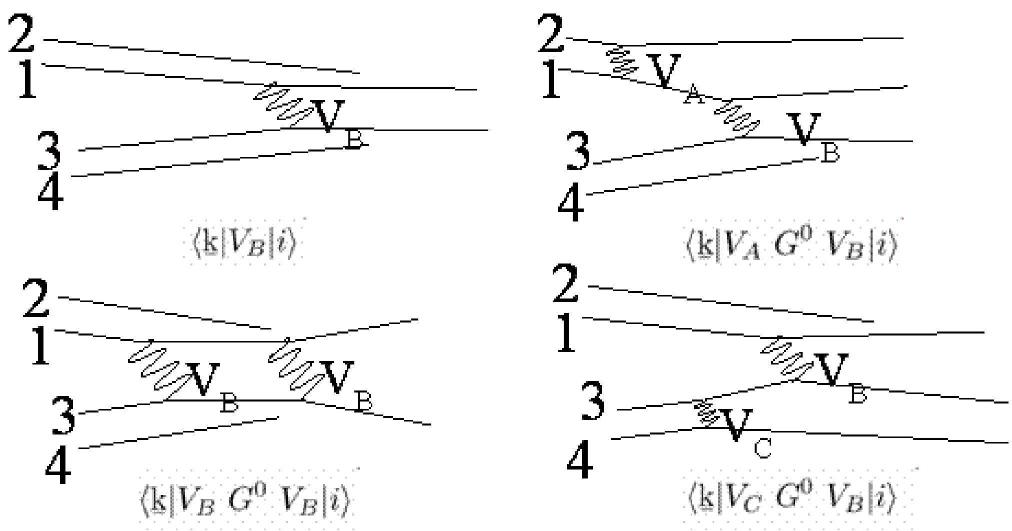

Figure 3: Four diagrams of the interactions represented by T_1 and T_2. The upper right interaction is T_1, while the other three are the three terms in T_2.

as particular interactions leading from the initial to the final state, whose contributions are then summed over. A diagrammatic representation of each of these interactions is provided in Figure 3.

Resolving the free Green's operator and inserting complete sets of states as necessary, we write

$$\langle \underline{k}|T_2|i\rangle = \langle \underline{k}|V_A\ G^0\ V_B|i\rangle + \langle \underline{k}|V_B\ G^0\ V_B|i\rangle + \langle \underline{k}|V_C\ G^0\ V_B|i\rangle \tag{113}$$

$$= \int d\underline{k}'d\underline{k}''\langle \underline{k}|V_A|\underline{k}'\rangle\langle \underline{k}'|G^0|\underline{k}''\rangle\langle \underline{k}''|V_B|i\rangle + \int d\underline{k}'d\underline{k}''\langle \underline{k}|V_B|\underline{k}'\rangle\langle \underline{k}'|G^0|\underline{k}''\rangle\langle \underline{k}''|V_B|i\rangle$$
$$+ \int d\underline{k}'d\underline{k}''\langle \underline{k}|V_C|\underline{k}'\rangle\langle \underline{k}'|G^0|\underline{k}''\rangle\langle \underline{k}''|V_B|i\rangle \tag{114}$$

$$= \int d\underline{k}'\frac{\langle \underline{k}|V_A|\underline{k}'\rangle}{z-T(\underline{k}')}\langle \underline{k}'|V_B|i\rangle + \int d\underline{k}'\frac{\langle \underline{k}|V_B|\underline{k}'\rangle}{z-T(\underline{k}')}\langle \underline{k}'|V_B|i\rangle + \int d\underline{k}'\frac{\langle \underline{k}|V_C|\underline{k}'\rangle}{z-T(\underline{k}')}\langle \underline{k}'|V_B|i\rangle \tag{115}$$

$$= -\frac{\lambda_A}{2\mu}\sum_m A_m(\mathbf{k_1})\int d\mathbf{k_1'}\frac{A_m^*(\mathbf{k_1'})}{z-T(k_1',k_2,K,P)}\langle \mathbf{k_1'},\mathbf{k_2},\mathbf{K},\mathbf{P}|\,V_B|i\rangle$$
$$-\frac{\lambda_B}{m}B(\frac{m}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2)\int d\mathbf{k_1'}\,d\mathbf{k_2'}\,d\mathbf{K}'\frac{B^*(\frac{m}{m+M}\mathbf{K'}-(\mathbf{k_2'}-\mathbf{k_1'})/2)}{z-T(k_1',k_2',K',P)}$$
$$\times\delta(\mathbf{k_1}+\mathbf{k_2}-\mathbf{k_1'}-\mathbf{k_2'})\delta(\frac{M}{m+M}\mathbf{K}+(\mathbf{k_2}-\mathbf{k_1})/2-\frac{M}{m+M}\mathbf{K'}-(\mathbf{k_2'}-\mathbf{k_1'})/2)\langle \mathbf{k_1'},\mathbf{k_2'},\mathbf{K'},\mathbf{P}|\,V_B|i\rangle$$
$$-\frac{\lambda_C}{2\mu}\sum_m C_m(\mathbf{k_2})\int d\mathbf{k_2'}\frac{C_m^*(\mathbf{k_2'})}{z-T(k_1,k_2',K,P)}\langle \mathbf{k_1},\mathbf{k_2'},\mathbf{K},\mathbf{P}|\,V_B|i\rangle\ . \tag{116}$$

We will tackle each term separately, and then sum them together with the matrix element of T_1 to arrive at the desired result. Inserting the result from the T_1 term to resolve the bracket, equation (101), the first piece is

$$\langle \underline{k}|V_A\ G^0\ V_B|i\rangle = -\frac{\lambda_A}{2\mu}\sum_m A_m(\mathbf{k_1})\int d\mathbf{k_1'}\frac{A_m^*(\mathbf{k_1'})}{z-T(k_1',k_2,K,P)}\langle \mathbf{k_1'},\mathbf{k_2},\mathbf{K},\mathbf{P}|\,V_B|i\rangle \tag{117}$$

$$= \frac{\lambda_A\lambda_B}{2m\mu}\sum_m A_m(\mathbf{k_1})\psi_C(\mathbf{k_2}-\frac{M}{m+M}(\mathbf{K_0}-\mathbf{K}))\delta(\mathbf{P_0}-\mathbf{P})$$
$$\times\int d\mathbf{k_1'}\frac{A_m^*(\mathbf{k_1'})}{z-T(k_1',k_2,K,P)}B(\frac{m}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1'})/2)B^*(\mathbf{K_0}-\frac{M}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1'})/2)$$
$$\times\,\psi_A(\mathbf{k_1'}+\frac{M}{m+M}(\mathbf{K_0}-\mathbf{K}))\ . \tag{118}$$

The goal now is to extract all information relating to helicity from the integral, and this is contained within terms that depend on the sign of m_A. So to continue we consider the integral alone, with the primes removed for clarity,

$$\int d\mathbf{k_1}\frac{A_m^*(\mathbf{k_1})}{z-T(k_1,k_2,K,P)}B(\frac{m}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2)B^*(\mathbf{K_0}-\frac{M}{m+M}\mathbf{K}-(\mathbf{k_2}-\mathbf{k_1})/2)$$
$$\times\,\psi_A(\mathbf{k_1}+\frac{M}{m+M}(\mathbf{K_0}-\mathbf{K}))$$
$$= 16N\sqrt{\frac{2\hbar}{\pi}}\hbar^l\int k_1^2dk_1\ \frac{k_1^l}{(k_1^2+\hbar^2\beta^2)^{l+1}(z-T(k_1,k_2,K,P))}$$

$$\times \int sin\theta_{k_1} d\theta_{k_1} d\phi_{k_1} \frac{|\mathbf{k_1} - \boldsymbol{\tau}|^l Y^*_{l,m}(\widehat{\mathbf{k_1}}) Y_{l,m_A}(\widehat{\mathbf{k_1} - \boldsymbol{\tau}})}{(4\hbar^2\gamma^2 + |\mathbf{k_1} - \boldsymbol{\rho}|^2)(4\hbar^2\gamma^2 + |\mathbf{k_1} - \boldsymbol{\sigma}|^2)(\alpha^2 + |\mathbf{k_1} - \boldsymbol{\tau}|^2)(\hbar^2\beta^2 + |\mathbf{k_1} - \boldsymbol{\tau}|^2)^{l+1}} \quad (119)$$

where

$$\boldsymbol{\rho} = \mathbf{k_2} - \frac{2m}{m+M}\mathbf{K} \ , \quad (120)$$

$$\boldsymbol{\sigma} = \mathbf{k_2} - 2\mathbf{K_0} + \frac{2M}{m+M}\mathbf{K} \ , \quad (121)$$

$$\boldsymbol{\tau} = -\frac{M}{m+M}(\mathbf{K_0} - \mathbf{K}) \ . \quad (122)$$

Each term of the form $|\mathbf{k_1} - \mathbf{p}|^\mathbf{2}$ can be expanded so that the dependence on $\mathbf{k_1}$ is separated out. The expansion is

$$|\mathbf{k_1} - \mathbf{p}|^\mathbf{2} = k_1^2 + p^2 - 2k_1 p \ cos\Theta_{k_1,p} \ , \quad (123)$$

where $\Theta_{k_1,p}$ is the angle between the two vectors and expands as

$$cos\Theta_{k_1,p} = cos\theta_{k_1} cos\theta_p + sin\theta_{k_1} sin\theta_p cos(\phi_{k_1} - \phi_p) \ . \quad (124)$$

We must also break apart the spherical harmonic into parts that depend separately on each of the vectors in its argument. This can be done using an expansion from the theories of bipolar spherical harmonics. The formalism is drawn from [10]. The expansion is

$$|\mathbf{k_1} - \boldsymbol{\tau}|^l Y_{l,m_A}(\widehat{\mathbf{k_1} - \boldsymbol{\tau}}) = \sqrt{4\pi(2l+1)!} \sum_{l_1+l_2=l} (-1)^{l_2} \frac{k_1^{l_1} \tau^{l_2}}{\sqrt{(2l_1+1)!(2l_2+1)!}} \times \left\{ \mathbf{Y_{l_1}}(\widehat{\mathbf{k_1}}) \otimes \mathbf{Y_{l_2}}(\widehat{\boldsymbol{\tau}}) \right\}_{l,m_A} , \quad (125)$$

where

$$\left\{ \mathbf{Y_{l_1}}(\widehat{\mathbf{k_1}}) \otimes \mathbf{Y_{l_2}}(\widehat{\boldsymbol{\tau}}) \right\}_{l,m_A} = \sum_{m_1,m_2} C^{l,m_A}_{l_1,m_1,l_2,m_2} Y_{l_1,m_1}(\widehat{\mathbf{k_1}}) Y_{l_2,m_2}(\widehat{\boldsymbol{\tau}}) \ , \quad (126)$$

and the $C^{l,m_A}_{l_1,m_1,l_2,m_2}$ are the Clebsch-Gordan Coefficients.

All together, the integral reduces to

$$= 16N\sqrt{\frac{2\hbar}{\pi}}\hbar^l \sqrt{4\pi(2l+1)!} \sum_{l_1+l_2=l} \frac{(-1)^{l_2}\tau^{l_2}}{\sqrt{(2l_1+1)!(2l_2+1)!}} \sum_{m_1,m_2} C^{l,m_A}_{l_1,m_1,l_2,m_2} Y_{l_2,m_2}(\widehat{\boldsymbol{\tau}}) \times \int dk_1 \ \frac{k_1^{l+2} k_1^{l_1}}{(k_1^2 + \hbar^2\beta^2)^{l+1}(z - T(k_1, k_2, K, P))}$$

$$\times \int sin\theta_{k_1} d\theta_{k_1} d\phi_{k_1} \frac{Y^*_{l,m}(\widehat{\mathbf{k_1}}) Y_{l_1,m_1}(\widehat{\mathbf{k_1}})}{(4\hbar^2\gamma^2 + |\mathbf{k_1} - \boldsymbol{\rho}|^2)(4\hbar^2\gamma^2 + |\mathbf{k_1} - \boldsymbol{\sigma}|^2)(\alpha^2 + |\mathbf{k_1} - \boldsymbol{\tau}|^2)(\hbar^2\beta^2 + |\mathbf{k_1} - \boldsymbol{\tau}|^2)^{l+1}} \quad (127)$$

To make some headway we must here reduce the generality of the problem somewhat by choosing values for the quantum numbers of interest. We will make the choice that the initial states ψ_A and ψ_C have $l = 1$, and that for ψ_A, $m_A = 1$. For ψ_C we choose $m_C = \pm 1$, with the positive value corresponding to the like helicity collision, and the negative value the unlike helicity collision. Inserting these values, the integration over the ϕ_{k_1} coordinate can be accomplished in order to make the dependence on the magnetic quantum number explicit. The integral can be done using contour integration, and the details are presented in Appendix B. The integration reduces the entire first piece to

$$\begin{aligned} \langle \underline{\mathrm{k}}|V_A G^0 V_B|i\rangle &= -\frac{\lambda_A\lambda_B}{2m\mu}\psi_C(\mathbf{k_2} - \frac{M}{M+m}(\mathbf{K_0}-\mathbf{K}))\delta(\mathbf{P_0}-\mathbf{P})\sum_m A_m(\mathbf{k_1}) \\ &\quad \times \sum_{l_1+l_2=1}\sum_{m_1,m_2} C^{1,1}_{l_1,m_1,l_2,m_2} Y_{l_2,m_2}(\,\hat{\boldsymbol{\tau}})I_{A,l_1,l_2,m,m_1}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\ , \end{aligned} \tag{128}$$

where the other remaining integrals have been denoted as I_{A,l_1,l_2,m,m_1} and the dependence on the various m quantum numbers is understood from the work in Appendix B. By inserting for the Clebsch Gordon coefficients of interest we can resolve the summations. The Clebsch Gordon coefficients vanish when the lower magnetic quantum numbers do not sum to give the upper magnetic quantum number, in this case enforcing $m_1 + m_2 = \pm 1$. So the coefficients of interest to the problem (both for work on this piece of the Born approximation and the fourth piece) are

$$C^{1,1}_{1,1,0,0} = 1\ , \tag{129}$$

$$C^{1,1}_{0,0,1,1} = 1\ , \tag{130}$$

$$C^{1,-1}_{1,-1,0,0} = 1\ , \tag{131}$$

$$C^{1,-1}_{0,0,1,-1} = 1\ . \tag{132}$$

Thankfully these are very simple coefficients. We need only the first two for this piece of the Born approximation. The sums are resolved as

$$\begin{aligned} \langle \underline{\mathrm{k}}|V_A G^0 V_B|i\rangle &= -\frac{\lambda_A\lambda_B}{2m\mu}\psi_C(\mathbf{k_2} - \frac{M}{M+m}(\mathbf{K_0}-\mathbf{K}))\delta(\mathbf{P_0}-\mathbf{P})\sum_m A_m(\mathbf{k_1}) \\ &\quad \times\left(\frac{1}{\sqrt{4\pi}}I_{A,1,0,m,1}(\mathbf{k_2},\mathbf{K},\mathbf{K_0}) + Y_{1,1}(\,\hat{\boldsymbol{\tau}})I_{A,0,1,m,0}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\right)\ . \end{aligned} \tag{133}$$

By the symmetries in the first and third pieces, I can immediately write the third term out. It is

$$\begin{aligned} \langle \underline{\mathrm{k}}|V_C\ G^0\ V_B|i\rangle &= \frac{\lambda_C\lambda_B}{2m\mu}\sum_m C_m(\mathbf{k_2})\psi_A(\mathbf{k_1} + \frac{M}{m+M}(\mathbf{K_0}-\mathbf{K}))\delta(\mathbf{P_0}-\mathbf{P}) \\ &\quad \times \int d\mathbf{k_2'}\frac{C^*_m(\mathbf{k_2'})}{z - T(k_1,k_2',K,P)}B(\frac{m}{m+M}\mathbf{K} - (\mathbf{k_2'}-\mathbf{k_1})/2)B^*(\mathbf{K_0} - \frac{M}{m+M}\mathbf{K} - (\mathbf{k_2'}-\mathbf{k_1})/2) \\ &\quad \times \psi_C(\mathbf{k_2'} - \frac{M}{m+M}(\mathbf{K_0}-\mathbf{K}))\ . \end{aligned} \tag{134}$$

The exact same series of steps as taken before apply to this piece, with the following substitutions made,

$$\begin{aligned}
A &\rightarrow C \ , && (135)\\
\mathbf{k_1} &\rightarrow \mathbf{k_2} \ , && (136)\\
\mathbf{k_2} &\rightarrow \mathbf{k_1} \ , && (137)\\
m_A &\rightarrow m_C \ , && (138)\\
\boldsymbol{\rho} &\rightarrow \boldsymbol{\zeta} = \mathbf{k_1} + \frac{2m}{M+m}\mathbf{K} \ , && (139)\\
\boldsymbol{\sigma} &\rightarrow \boldsymbol{\xi} = \mathbf{k_1} - \frac{2M}{M+m}\mathbf{K} + 2\mathbf{K_0} \ , && (140)\\
\boldsymbol{\tau} &\rightarrow \boldsymbol{\chi} = \frac{M}{M+m}(\mathbf{K_0} - \mathbf{K}) \ . && (141)
\end{aligned}$$

With these steps taken and the same integration from Appendix B as used before, the third piece reduces to

$$\begin{aligned}
\langle \underline{\mathrm{k}} | V_C G^0 V_B | i \rangle &= -\frac{\lambda_C \lambda_B}{2m\mu} \psi_A(\mathbf{k_1} + \frac{M}{M+m}(\mathbf{K_0} - \mathbf{K}))\delta(\mathbf{P_0} - \mathbf{P}) \\
&\quad \times \sum_m C_m(\mathbf{k_2}) \bigg(\frac{1}{\sqrt{4\pi}} I_{C,1,0,m,\pm 1}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) \\
&\qquad + Y_{1,\pm 1}(\widehat{\boldsymbol{\chi}}) I_{C,0,1,m,0}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) \bigg) \qquad (142)
\end{aligned}$$

Now, we calculate the second piece, which is

$$\begin{aligned}
\langle \underline{\mathrm{k}} | V_B \ G^0 \ V_B | i \rangle &= -\frac{\lambda_B}{m} \ B(\frac{m}{m+M}\mathbf{K} - (\mathbf{k_2} - \mathbf{k_1})/2) \int d\mathbf{k'_1} \, d\mathbf{k'_2} \, d\mathbf{K'} \frac{B^*(\frac{m}{m+M}\mathbf{K'} - (\mathbf{k'_2} - \mathbf{k'_1})/2)}{z - T(k'_1, k'_2, K', P)} \\
&\quad \times \delta(\mathbf{k_1} + \mathbf{k_2} - \mathbf{k'_1} - \mathbf{k'_2}) \delta(\frac{M}{m+M}\mathbf{K} + (\mathbf{k_2} - \mathbf{k_1})/2 - \frac{M}{m+M}\mathbf{K'} - (\mathbf{k'_2} - \mathbf{k'_1})/2) \\
&\quad \times \langle \mathbf{k'_1}, \mathbf{k'_2}, \mathbf{K'}, \mathbf{P} | \mathbf{V_B} | \mathbf{i} \rangle \qquad (143) \\
&= -\frac{\lambda_B}{m} \ B(\frac{m}{m+M}\mathbf{K} - (\mathbf{k_2} - \mathbf{k_1})/2) \\
&\quad \times \int d\mathbf{K'} \frac{B^*(\mathbf{K'} - \frac{M}{m+M}\mathbf{K} - (\mathbf{k_2} - \mathbf{k_1})/2)}{z - \frac{P^2 + 4K'^2}{4(m+M)} - \frac{k_1^2 + k_2^2}{2\mu} - \frac{\frac{M}{m+M}(K' - K)(\frac{M}{m+M}(K' - K) + (k_1 - k_2)/2)}{\mu}} \\
&\quad \times \langle \mathbf{k_1} + \frac{M}{m+M}(\mathbf{K'} - \mathbf{K}), \mathbf{k_2} - \frac{M}{m+M}(\mathbf{K'} - \mathbf{K}), \mathbf{K'}, \mathbf{P} | \, V_B | i \rangle \ . \qquad (144)
\end{aligned}$$

Using equation (101) with the appropriate substitutions to resolve the bracket,

$$\begin{aligned}
\langle \underline{\mathrm{k}} | V_B \ G^0 \ V_B | i \rangle &= \frac{\lambda_B^2}{m^2} \ B(\frac{m}{m+M}\mathbf{K} - (\mathbf{k_2} - \mathbf{k_1})/2) B^*(\mathbf{K_0} - \frac{M}{m+M}\mathbf{K} - (\mathbf{k_2} - \mathbf{k_1})/2) \\
&\quad \times \delta(\mathbf{P_0} - \mathbf{P}) \psi_A(\mathbf{k_1} + \frac{M}{m+M}(\mathbf{K_0} - \mathbf{K})) \psi_C(\mathbf{k_2} - \frac{M}{m+M}(\mathbf{K_0} - \mathbf{K})) \\
&\quad \times \int d\mathbf{K'} \frac{|B(\mathbf{K'} - \frac{M}{m+M}\mathbf{K} - (\mathbf{k_2} - \mathbf{k_1})/2)|^2}{z - \frac{P^2 + 4K'^2}{4(m+M)} - \frac{k_1^2 + k_2^2}{2\mu} - \frac{\frac{M}{m+M}(K' - K)(\frac{M}{m+M}(K' - K) + (k_1 - k_2)/2)}{\mu}} \qquad (145)
\end{aligned}$$

With all the pieces in hand, we are ready to construct the T matrix element. We first make the substitutions

$$\mathbf{x} = \mathbf{k_1} + \frac{M}{m+M}(\mathbf{K_0} - \mathbf{K}) \ , \tag{146}$$

$$\mathbf{y} = \mathbf{k_2} - \frac{M}{m+M}(\mathbf{K_0} - \mathbf{K}) \ . \tag{147}$$

(148)

To clarify the asymmetry calculation to follow, we combine all quantities that do not depend on the m quantum numbers into a function for each piece calculated above. In doing so, we use the separation of initial state wavefunctions ψ into their "radial " and angular parts. The definitions of these m independent quantities are then

$$F_1(\mathbf{k_1},\mathbf{k_2},\mathbf{K},\mathbf{K_0}) = -\frac{\lambda_B}{m}\ B(\frac{m}{m+M}\mathbf{K} - (\mathbf{k_2}-\mathbf{k_1})/2)B^*(\mathbf{K_0} - \frac{M}{m+M}\mathbf{K} - (\mathbf{k_2}-\mathbf{k_1})/2) \times R_A(x)R_C(y) \ , \tag{149}$$

$$F_2(\mathbf{k_2},\mathbf{K},\mathbf{K_0}) = -\frac{\lambda_A\lambda_B}{2m\mu}R_C(y) \ , \tag{150}$$

$$F_3(\mathbf{k_1},\mathbf{k_2}\mathbf{K},\mathbf{K_0}) = \frac{\lambda_B^2}{m^2}\ B(\frac{m}{m+M}\mathbf{K} + (\mathbf{k_2}-\mathbf{k_1})/2)B^*(\mathbf{K_0} - \frac{M}{m+M}\mathbf{K} - (\mathbf{k_2}-\mathbf{k_1})/2) \times R_A(x)R_C(x) \times \int d\mathbf{K'}\frac{|B(\mathbf{K'} - \frac{M}{m+M}\mathbf{K} - (\mathbf{k_2}-\mathbf{k_1})/2)|^2}{z - \frac{P^2+4K'^2}{4(m+M)} - \frac{k_1^2+k_2^2}{2\mu} - \frac{\frac{M}{m+M}(K'-K)(\frac{M}{m+M}(K'-K)+(k_1-k_2)/2)}{\mu}} \tag{151}$$

$$F_4(\mathbf{k_2},\mathbf{K},\mathbf{K_0}) = -\frac{\lambda_C\lambda_B}{2m\mu}R_A(x) \ . \tag{152}$$

We at this point again drop the delta function $\delta(\mathbf{P_0} - \mathbf{P})$ from further calculations. Taking the definitions above, the total on shell T matrix is

$$\langle \underline{\mathrm{k}}|T_1+T_2|i\rangle = \langle \underline{\mathrm{k}}|V_B|i\rangle + \langle \underline{\mathrm{k}}|V_A\ G^0\ V_B|i\rangle + \langle \underline{\mathrm{k}}|V_B\ G^0\ V_B|i\rangle + \langle \underline{\mathrm{k}}|V_C\ G^0\ V_B|i\rangle \tag{153}$$

$$\begin{aligned} = {} & Y_{1,1}(\widehat{\mathbf{x}})Y_{1,\pm1}(\widehat{\mathbf{y}})F_1(\mathbf{k_1},\mathbf{k_2},\mathbf{K},\mathbf{K_0}) \\ & + Y_{1,\pm1}(\widehat{\mathbf{y}})F_2(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m A_m(\mathbf{k_1})\left(\frac{1}{\sqrt{4\pi}}I_{A,1,0,m,1}(\mathbf{k_2},\mathbf{K},\mathbf{K_0}) + Y_{1,1}(\widehat{\boldsymbol{\tau}})I_{A,0,1,m,0}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\right) \\ & + Y_{1,1}(\widehat{\mathbf{x}})Y_{1,\pm1}(\widehat{\mathbf{y}})F_3(\mathbf{k_1},\mathbf{k_2},\mathbf{K},\mathbf{K_0}) \\ & + Y_{1,1}(\widehat{\mathbf{x}})F_4(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m C_m(\mathbf{k_2})\left(\frac{1}{\sqrt{4\pi}}I_{C,1,0,m,\pm1}(\mathbf{k_1},\mathbf{K},\mathbf{K_0}) \right. \\ & \left. + Y_{1,\pm1}(\widehat{\boldsymbol{\chi}})I_{C,0,1,m,0}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})\right) \ . \end{aligned} \tag{154}$$

5.7 Helicity Asymmetry in the Second Born

We recall that the differential cross section $\frac{D\sigma}{D\Omega}$ discussed earlier is proportional to the square of the T matrix element, which is denoted for brevity as $|t|^2$ and is

$$
\begin{aligned}
|t|^2 \;=\;& |Y_{1,1}(\widehat{\mathbf{x}})Y_{1,\pm1}(\widehat{\mathbf{y}})|^2(|F_1(\mathbf{k_1},\mathbf{K},\mathbf{K_0})|^2+|F_3(\mathbf{k_1},\mathbf{k_2},\mathbf{K},\mathbf{K_0})|^2)\\
&+\left|Y_{1,\pm1}(\widehat{\mathbf{y}})F_2(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m A_m(\mathbf{k_1})\left(\frac{1}{\sqrt{4\pi}})I_{A,1,0,m,1,0}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})+Y_{1,1}(\widehat{\boldsymbol{\tau}})I_{A,0,1,m,0,1}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\right)\right|^2\\
&+\left|Y_{1,1}(\widehat{\mathbf{x}})F_4(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m C_m(\mathbf{k_2})\left(\frac{1}{\sqrt{4\pi}}I_{C,1,0,m,\pm1}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})+Y_{1,\pm1}(\widehat{\boldsymbol{\chi}})I_{C,0,1,m,0}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})\right)\right|^2\\
&+Y_{1,1}(\widehat{\mathbf{x}})Y_{1,\pm1}(\widehat{\mathbf{y}})F_1(\mathbf{k_1},\mathbf{K},\mathbf{K_0})\\
&\times\left\{Y_{1,\pm1}(\widehat{\mathbf{y}})F_2(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m A_m(\mathbf{k_1})\left(\frac{1}{\sqrt{4\pi}}I_{A,1,0,m,1}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})+Y_{1,1}(\widehat{\boldsymbol{\tau}})I_{A,0,1,m,0}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\right)\right.\\
&+Y_{1,1}(\widehat{\mathbf{x}})Y_{1,\pm1}(\widehat{\mathbf{y}})F_3(\mathbf{k_1},\mathbf{k_2},\mathbf{K},\mathbf{K_0})\\
&\left.+Y_{1,1}(\widehat{\mathbf{x}})F_4(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m C_m(\mathbf{k_2})\left(\frac{1}{\sqrt{4\pi}}I_{C,1,0,m,\pm1}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})+Y_{1,\pm1}(\widehat{\boldsymbol{\chi}})I_{C,0,1,m,0}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})\right)\right\}^*\\
&+Y_{1,\pm1}(\widehat{\mathbf{y}})F_2(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m A_m(\mathbf{k_1})\left(\frac{1}{\sqrt{4\pi}}I_{A,1,0,m,1,0}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})+Y_{1,1}(\widehat{\boldsymbol{\tau}})I_{A,0,1,m,0,1}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\right)\\
&\times\left\{Y_{1,1}(\widehat{\mathbf{x}})Y_{1,\pm1}(\widehat{\mathbf{y}})(F_1(\mathbf{k_1},\mathbf{K},\mathbf{K_0})+F_3(\mathbf{k_1},\mathbf{k_2},\mathbf{K},\mathbf{K_0}))\right.\\
&\left.+Y_{1,1}(\widehat{\mathbf{x}})F_4(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m C_m(\mathbf{k_2})\left(\frac{1}{\sqrt{4\pi}}I_{C,1,0,m,\pm1}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})+Y_{1,\pm1}(\widehat{\boldsymbol{\chi}})I_{C,0,1,m,0}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})\right)\right\}^*\\
&+Y_{1,1}(\widehat{\mathbf{x}})Y_{1,\pm1}(\widehat{\mathbf{y}})F_3(\mathbf{k_1},\mathbf{k_2},\mathbf{K},\mathbf{K_0})\left\{Y_{1,1}(\widehat{\mathbf{x}})Y_{1,\pm1}(\widehat{\mathbf{y}})F_1(\mathbf{k_1},\mathbf{K},\mathbf{K_0})\right.\\
&+Y_{1,\pm1}(\widehat{\mathbf{y}})F_2(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m A_m(\mathbf{k_1})\left(\frac{1}{\sqrt{4\pi}}I_{A,1,0,m,1,0}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})+Y_{1,1}(\widehat{\boldsymbol{\tau}})I_{A,0,1,m,0,1}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\right)\\
&\left.+Y_{1,1}(\widehat{\mathbf{x}})F_4(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m C_m(\mathbf{k_2})\left(\frac{1}{\sqrt{4\pi}}I_{C,1,0,m,\pm1}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})+Y_{1,\pm1}(\widehat{\boldsymbol{\chi}})I_{C,0,1,m,0}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})\right)\right\}^*\\
&+Y_{1,1}(\widehat{\mathbf{x}})F_4(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m C_m(\mathbf{k_2})\left(\frac{1}{\sqrt{4\pi}}I_{C,1,0,m,\pm1}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})+Y_{1,\pm1}(\widehat{\boldsymbol{\chi}})I_{C,0,1,m,0}(\mathbf{k_1},\mathbf{K},\mathbf{K_0})\right)\\
&\times\left\{Y_{1,\pm1}(\widehat{\mathbf{y}})F_2(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\sum_m A_m(\mathbf{k_1})\left(\frac{1}{\sqrt{4\pi}}I_{A,1,0,m,1,0}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})+Y_{1,1}(\widehat{\boldsymbol{\tau}})I_{A,0,1,m,0,1}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\right)\right.\\
&\left.+Y_{1,1}(\widehat{\mathbf{x}})Y_{1,\pm1}(\widehat{\mathbf{y}})(F_1(\mathbf{k_1},\mathbf{K},\mathbf{K_0})+F_3(\mathbf{k_1},\mathbf{k_2},\mathbf{K},\mathbf{K_0}))\right\}^*. \qquad (155)
\end{aligned}
$$

With this we can calculate the asymmetry, where for like helicity collisions, we use the + case in each $\pm$ and for unlike helicity we will use the $-$ case in each $\pm$. All of the terms that have no helicity dependence are eliminated, and so the asymmetry is

$$
ASM = \frac{\frac{D\sigma}{D\Omega}_{like}-\frac{D\sigma}{D\Omega}_{unlike}}{\frac{D\sigma}{D\Omega}_{like}+\frac{D\sigma}{D\Omega}_{unlike}} \qquad (156)
$$

$$
= \frac{|t|^2_{like}-|t|^2_{unlike}}{|t|^2_{like}+|t|^2_{unlike}} \qquad (157)
$$

$$
\begin{aligned}
= \quad & \Bigg\{ |Y_{1,1}(\widehat{\mathbf{x}}) F_4(\mathbf{k_2}, \mathbf{K}, \mathbf{K_0})|^2 \\
& \times \Bigg[\Bigg| \sum_m C_m(\mathbf{k_2}) \Bigg(\frac{1}{\sqrt{4\pi}} I_{C,1,0,m,1}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) + Y_{1,1}(\widehat{\boldsymbol{\chi}}) I_{C,0,1,m,0}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) \Bigg) \Bigg|^2 \\
& - \Bigg| \sum_m C_m(\mathbf{k_2}) \Bigg(\frac{1}{\sqrt{4\pi}} I_{C,1,0,m,-1}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) + Y_{1,-1}(\widehat{\boldsymbol{\chi}}) I_{C,0,1,m,0}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) \Bigg) \Bigg|^2 \Bigg] \\
& + 2Re \Bigg[Y_{1,1}(\widehat{\mathbf{x}}) F_4(\mathbf{k_2}, \mathbf{K}, \mathbf{K_0}) \\
& \times \sum_m C_m(\mathbf{k_2}) \Bigg(\frac{1}{\sqrt{4\pi}} I_{C,1,0,m,1}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) + Y_{1,1}(\widehat{\boldsymbol{\chi}}) I_{C,0,1,m,0}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) \Bigg) \\
& \times \Bigg\{ Y_{1,1}(\widehat{\mathbf{y}}) F_2(\mathbf{k_2}, \mathbf{K}, \mathbf{K_0}) \\
& \times \sum_m A_m(\mathbf{k_1}) \Bigg(\frac{1}{\sqrt{4\pi}} I_{A,1,0,m,1,0}(\mathbf{k_2}, \mathbf{K}, \mathbf{K_0}) + Y_{1,1}(\widehat{\boldsymbol{\tau}}) I_{A,0,1,m,0,1}(\mathbf{k_2}, \mathbf{K}, \mathbf{K_0}) \Bigg) \\
& + Y_{1,1}(\widehat{\mathbf{x}}) Y_{1,1}(\widehat{\mathbf{y}}) (F_1(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) + F_3(\mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{K_0})) \Bigg\}^* \\
& - Y_{1,1}(\widehat{\mathbf{x}}) F_4(\mathbf{k_2}, \mathbf{K}, \mathbf{K_0}) \\
& \times \sum_m C_m(\mathbf{k_2}) \Bigg(\frac{1}{\sqrt{4\pi}} I_{C,1,0,m,\pm 1}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) + Y_{1,\pm 1}(\widehat{\boldsymbol{\chi}}) I_{C,0,1,m,0}(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) \Bigg) \\
& \times \Bigg\{ Y_{1,-1}(\widehat{\mathbf{y}}) F_2(\mathbf{k_2}, \mathbf{K}, \mathbf{K_0}) \\
& \times \sum_m A_m(\mathbf{k_1}) \Bigg(\frac{1}{\sqrt{4\pi}} I_{A,1,0,m,1,0}(\mathbf{k_2}, \mathbf{K}, \mathbf{K_0}) + Y_{1,1}(\widehat{\boldsymbol{\tau}}) I_{A,0,1,m,0,1}(\mathbf{k_2}, \mathbf{K}, \mathbf{K_0}) \Bigg) \\
& + Y_{1,1}(\widehat{\mathbf{x}}) Y_{1,-1}(\widehat{\mathbf{y}}) (F_1(\mathbf{k_1}, \mathbf{K}, \mathbf{K_0}) + F_3(\mathbf{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{K_0})) \Bigg\}^* \Bigg] \Bigg\} \\
& \times \Bigg(|t|^2_{like} + |t|^2_{unlike} \Bigg)^{-1} . \qquad (158)
\end{aligned}
$$

It is not obvious that the above asymmetry vanishes. In fact, given the character of the various terms, especially of the various I_A and I_C terms, it would be very surprising if this asymmetry was equal to zero. However, a rigorous investigation involving numerical approaches would be required to calculate a few sample asymmetries and prove that the above asymmetry does not vanish in an unexpected way.

6 Conclusions

There are two key conclusions to be had from the four body helicity asymmetry calculation. The first is that there is in general a helicity asymmetry in the differential cross section of the breakup scattering process. The second is that this asymmetry occurs at the second order of the Born approximation.

With the assumption that our model can give us qualitative predictions about the collision of polarized protons, we see that we expect a helicity asymmetry if the proton is made up of rotating constituents. In addition, we expect that this effect should be small compared to first order effects.

This prediction is complicated by the structure of the proton. For instance, quarks of different flavors may have angular momentum directed in opposite directions, and the sea of virtual quarks may not have any preferred direction at all. If any effect is to be found, collisions between different flavors of quarks and between the valence and sea quarks must be separated out and investigated independently. In addition to this are the gluons, which themselves may have angular momentum, and make for even more possible interactions. On the other hand, if this separation can be accomplished, there is the possibility of using helicity asymmetries to probe the detailed structure of the proton.

A group here at the University of New Mexico, led by Doug Fields, has investigated a specific kind of helicity asymmetry in polarized proton-proton collisions, and these results are especially interesting in light of their research [2]. The data used is that from the PHENIX detector at the RHIC particle collider. This group has studied the momentum transverse to the beam direction in both like and unlike helicity collisions. Due to conservation of momentum, it is not expected that the jets produced in a hard scattering collision will have any total momentum transverse to the beam direction. However, various effects contribute to an intrinsic transverse momentum in the jets. As suggested by the semi-classical studies discussed earlier, the angular momentum of the partons is one possible source. However, Fields and his group have found that there is no asymmetry in the transverse momentum. This serves to confirm the prediction that any asymmetry would be a small effect, and may even show that there is no net orbital angular momentum present in the proton.

Clearly we must be careful in making physical predictions. The nonrelativistic scattering process described here is obviously very different from the relativistic, chromodynamical processes governing the collision of two protons. Also, in order to make real statements about the predicted asymmetry in transverse momentum, the cross sections presented in equation (158) must be integrated over the variables that do not involve the transverse momentum, an effort which would itself demand another study of similar length. Finally, all effects of spin have been neglected from this study. The potentials selected were not spin dependent, and so neglecting spin makes no difference in the results; however, since the strong interaction is spin dependent, further studies might use spin dependent potentials to better model hard proton-proton scattering.

7 References

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A The Coordinates of the Four Body Problem

In order to find the matrix element of V_B in terms of the set of relative momenta $\underline{\mathrm{k}}$, we first consider the matrix element of V_B in position space in terms of the cartesian coordinates. V_B acts between particles 1 and 3. The potential is nonlocal, acts on the coordinates $\mathbf{x_3} - \mathbf{x_1}$, but ignores the coordinates $\mathbf{x_2}$ and $\mathbf{x_4}$. It must also leave the center of mass coordinate between particles 1 and 3, so that the combination $(\mathbf{x_1} + \mathbf{x_3})/2$ is ignored as well. Taking all this into account, the matrix element in position space must look like

$$\begin{aligned} \langle \underline{\mathrm{x}} | V_B | \underline{\mathrm{x}}' \rangle &= \delta(\mathbf{x_2} - \mathbf{x_2'})\delta(\mathbf{x_4} - \mathbf{x_4'})\delta(\frac{\mathbf{x_1} + \mathbf{x_3}}{2} - \frac{\mathbf{x_1'} + \mathbf{x_3'}}{2}) \\ &\quad \times V_B(\mathbf{x_3} - \mathbf{x_1}, \mathbf{x_3'} - \mathbf{x_1'}) \ . \end{aligned} \tag{159}$$

The next step is to consider the matrix element of V_B in the usual set of momenta for each of the four particles, $\{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}\}$. This is given with the help of the position space representation,

$$\langle \underline{\mathrm{p}} | V_B | \underline{\mathrm{p}}' \rangle = \int d\underline{\mathrm{x}} d\underline{\mathrm{x}}' \langle \underline{\mathrm{p}} | \underline{\mathrm{x}} \rangle \langle \underline{\mathrm{x}} | V_B | \underline{\mathrm{x}}' \rangle \langle \underline{\mathrm{x}}' | \underline{\mathrm{p}}' \rangle \tag{160}$$

$$\begin{aligned} &= (2\pi\hbar)^{-9} \int d\underline{\mathrm{x}} d\underline{\mathrm{x}}' e^{i(\mathbf{p_1}\cdot\mathbf{x_1} + \mathbf{p_2}\cdot\mathbf{x_2} + \mathbf{p_3}\cdot\mathbf{x_3} + \mathbf{p_4}\cdot\mathbf{x_4})/\hbar} \\ &\quad \times \delta(\mathbf{x_2} - \mathbf{x_2'})\delta(\mathbf{x_4} - \mathbf{x_4'})\delta(\frac{\mathbf{x_1} + \mathbf{x_3} - \mathbf{x_1'} - \mathbf{x_3'}}{2}) V_B(\mathbf{x_3} - \mathbf{x_1}, \mathbf{x_3'} - \mathbf{x_1'}) \\ &\quad \times e^{-i(\mathbf{p_1'}\cdot\mathbf{x_1'} + \mathbf{p_2'}\cdot\mathbf{x_2'} + \mathbf{p_3'}\cdot\mathbf{x_3'} + \mathbf{p_4'}\cdot\mathbf{x_4'})/\hbar} \ . \end{aligned} \tag{161}$$

This simplifies more readily with a simple change of variables

$$\mathbf{y} = \mathbf{x_3} - \mathbf{x_1} \ , \tag{162}$$

$$\mathbf{z} = \frac{\mathbf{x_1} + \mathbf{x_3}}{2} \ , \tag{163}$$

$$\mathbf{x_1} = \mathbf{z} - \mathbf{y}/2 \ , \tag{164}$$

$$\mathbf{x_3} = \mathbf{z} + \mathbf{y}/2 \ , \tag{165}$$

so that the matrix element is now

$$
\begin{aligned}
\langle \underline{\mathrm{p}}|V_B|\underline{\mathrm{p}}'\rangle &= (2\pi\hbar)^{-9}\int d\mathbf{x_2}\,d\mathbf{x_4}\,d\mathbf{y}\,d\mathbf{z}d\mathbf{x'_2}\,d\mathbf{x'_4}\,d\mathbf{y'}\,d\mathbf{z'} \\
&\quad\times e^{i(\mathbf{p_2}\cdot\mathbf{x_2}+\mathbf{p_4}\cdot\mathbf{x_4}+(\mathbf{p_1}+\mathbf{p_3})\cdot\mathbf{z}+(\mathbf{p_3}-\mathbf{p_1})/2\cdot\mathbf{y})/\hbar} \\
&\quad\times \delta(\mathbf{x_2}-\mathbf{x'_2})\delta(\mathbf{x_4}-\mathbf{x'_4})\delta(\mathbf{z}-\mathbf{z'})\,V_B(\mathbf{y},\mathbf{y'}) \\
&\quad\times e^{-i(\mathbf{p'_2}\cdot\mathbf{x'_2}+\mathbf{p'_4}\cdot\mathbf{x'_4}+(\mathbf{p'_1}+\mathbf{p'_3})\cdot\mathbf{z'}+(\mathbf{p'_3}-\mathbf{p'_1})/2\cdot\mathbf{y'})/\hbar} \qquad (166)\\
&= (2\pi\hbar)^{-9}\int d\mathbf{x_2}\,d\mathbf{x_4}\,d\mathbf{y}\,d\mathbf{z}d\mathbf{y'}\;V_B(\mathbf{y},\mathbf{y'}) \\
&\quad\times e^{i((\mathbf{p_2}-\mathbf{p'_2})\cdot\mathbf{x_2}+(\mathbf{p_4}-\mathbf{p'_4})\cdot\mathbf{x_4})/\hbar} \\
&\quad\times e^{i((\mathbf{p_1}+\mathbf{p_3}-\mathbf{p'_1}-\mathbf{p'_3})\cdot\mathbf{z}+(\mathbf{p_3}-\mathbf{p_1}-\mathbf{p'_3}+\mathbf{p'_1})/2\cdot\mathbf{y})/\hbar} \qquad (167)\\
&= (2\pi\hbar)^{-3}\delta(\mathbf{p_2}-\mathbf{p'_2})\delta(\mathbf{p_4}-\mathbf{p'_4})\delta(\mathbf{p_1}+\mathbf{p_3}-\mathbf{p'_1}-\mathbf{p'_3}) \\
&\quad\times\int d\mathbf{y}\,d\mathbf{y'}\,e^{i((p_3-p_1-p'_3+p'_1)/2\cdot\mathbf{y})/\hbar}\,V_B(\mathbf{y},\mathbf{y'})\ . \qquad (168)
\end{aligned}
$$

The last integral we recognize as the Fourier transform of $V_B(\mathbf{y},\mathbf{y'})$, which we will denote here simply by $V_B(\frac{\mathbf{p_3}-\mathbf{p_1}}{2},\frac{\mathbf{p'_3}-\mathbf{p'_1}}{2})$. This also absorbs the extra factors of two, π and $\hbar$. Section 5.3 carries out this process explicitly. All together then the matrix element is

$$
\begin{aligned}
\langle \underline{\mathrm{p}}|V_B|\underline{\mathrm{p}}'\rangle &= \delta(\mathbf{p_2}-\mathbf{p'_2})\delta(\mathbf{p_4}-\mathbf{p'_4})\delta(\mathbf{p_1}+\mathbf{p_3}-\mathbf{p'_1}-\mathbf{p'_3}) \\
&\quad\times V_B(\frac{\mathbf{p_3}-\mathbf{p_1}}{2},\frac{\mathbf{p'_3}-\mathbf{p'_1}}{2})\ . \qquad (169)
\end{aligned}
$$

The final step is of course to move from these momenta coordinates into the relative momenta. For this we note that the relations between the $\underline{\mathrm{p}}$ momenta and the $\underline{\mathrm{k}}$ momenta are

$$
\begin{aligned}
\mathbf{p_1} &= -\mathbf{k_1}+\frac{m}{2(m+M)}(\mathbf{P}-2\mathbf{K})\ , \qquad (170)\\
\mathbf{p_2} &= \mathbf{k_1}+\frac{M}{2(m+M)}(\mathbf{P}-2\mathbf{K})\ , \qquad (171)\\
\mathbf{p_3} &= -\mathbf{k_2}+\frac{m}{2(m+M)}(\mathbf{P}+2\mathbf{K})\ , \qquad (172)\\
\mathbf{p_4} &= \mathbf{k_2}+\frac{M}{2(m+M)}(\mathbf{P}+2\mathbf{K})\ . \qquad (173)
\end{aligned}
$$

Next, we note that by definition the bracket between the $\underline{\mathrm{k}}$ momenta and the $\underline{\mathrm{p}}$ momenta just generates the delta functions that enforce the above relationships, so that the expansion of $\underline{\mathrm{k}}$ into $\underline{\mathrm{p}}$ momenta looks like

$$
\begin{aligned}
|\mathbf{k_1},\mathbf{k_2},\mathbf{K},\mathbf{P}\rangle &= \int d\underline{\mathrm{p}}\ \delta(\mathbf{p_1}+\mathbf{k_1}-\frac{m}{2(m+M)}(\mathbf{P}-2\mathbf{K})) \\
&\quad\times\delta(\mathbf{p_2}-\mathbf{k_1}-\frac{M}{2(m+M)}(\mathbf{P}-2\mathbf{K})) \\
&\quad\times\delta(\mathbf{p_3}+\mathbf{k_2}-\frac{m}{2(m+M)}(\mathbf{P}+2\mathbf{K}))
\end{aligned}
$$

$$\times\delta(\mathbf{p_4} - \mathbf{k_2} - \frac{M}{2(m+M)}(\mathbf{P}+2\mathbf{K}))$$
$$\times|\mathbf{p_1},\mathbf{p_2},\mathbf{p_3},\mathbf{p_4}\rangle\ . \quad (174)$$

Using these expansions, it can be seen that the desired matrix element of V_B in terms of the $\underline{\mathrm{k}}$ momenta is the unsightly expression

$$\begin{aligned}\langle\underline{\mathrm{k}}|V_B|\underline{\mathrm{k}}'\rangle &= \delta(\mathbf{k_1} + \frac{M}{2(m+M)}(\mathbf{P}-2\mathbf{K}) - \mathbf{k_1'} - \frac{M}{2(m+M)}(\mathbf{P'}-2\mathbf{K'}))\\ &\times\delta(\mathbf{k_2} + \frac{M}{2(m+M)}(\mathbf{P}+2\mathbf{K}) - \mathbf{k_2'} - \frac{M}{2(m+M)}(\mathbf{P'}+2\mathbf{K'})\\ &\times\delta(-\mathbf{k_1} - \mathbf{k_2} + \frac{m}{m+M}\mathbf{P} + \mathbf{k_1'} + \mathbf{k_2'} - \frac{m}{m+M}\mathbf{P'})\\ &\times V_B(\frac{m}{m+M}\mathbf{K} - \frac{\mathbf{k_2}-\mathbf{k_1}}{2}, \frac{m}{m+M}\mathbf{K'} - \frac{\mathbf{k_2'}-\mathbf{k_1'}}{2})\ . \end{aligned} \quad (175)$$

There is only a single step separating this from the form quoted in section 5.2, equation (**??**). We recognize at this point that the delta functions given here enforce the same set of relationships as

$$\begin{aligned}&\delta(\frac{M}{m+M}\mathbf{K} - (\mathbf{k_2}-\mathbf{k_1})/2 - \frac{M}{m+M}\mathbf{K'} + (\mathbf{k_2'}-\mathbf{k_1'})/2)\\ &\times\delta(\mathbf{k_1}+\mathbf{k_2}-\mathbf{k_1'}-\mathbf{k_2'})\delta(\mathbf{P}-\mathbf{P'})\ ,\end{aligned} \quad (176)$$

so we replace the first set of delta functions with this cleaner, more physically relevant set of delta functions. This gives finally

$$\begin{aligned}\langle\underline{\mathrm{k}}|V_B|\underline{\mathrm{k}}'\rangle &= \delta(\frac{M}{m+M}\mathbf{K} - (\mathbf{k_2}-\mathbf{k_1})/2 - \frac{M}{m+M}\mathbf{K'} + (\mathbf{k_2'}-\mathbf{k_1'})/2)\\ &\times\delta(\mathbf{k_1}+\mathbf{k_2}-\mathbf{k_1'}-\mathbf{k_2'})\delta(\mathbf{P}-\mathbf{P'})\\ &\times V_B(\frac{m}{m+M}\mathbf{K} - \frac{\mathbf{k_2}-\mathbf{k_1}}{2}, \frac{m}{m+M}\mathbf{K'} - \frac{\mathbf{k_2'}-\mathbf{k_1'}}{2})\ .\end{aligned} \quad (177)$$

B The Contour Integral

The integral of interest, (119), with $l=1$ and $m_A=1$ is

$$\begin{aligned}&16N\sqrt{\frac{2\hbar}{\pi}}\hbar\sqrt{24\pi}\sum_m\sum_{l_1+l_2=1}\frac{(-1)^{l_2}\tau^{l_2}}{\sqrt{(2l_1+1)!(2l_2+1)!}}\sum_{m_1,m_2}C^{1,1}_{l_1,m_1,l_2,m_2}Y_{l_2,m_2}(\widehat{\boldsymbol{\tau}})\\ &\times\int dk_1\ \frac{k_1^{l+2}k_1^{l_1}}{(k_1^2+\hbar^2\beta^2)^2(z-T(k_1,k_2,K,P)}\\ &\times\int sin\theta_{k_1}d\theta_{k_1}d\phi_{k_1}\frac{Y^*_{1,m}(\widehat{\mathbf{k_1}})Y_{l_1,m_1}(\widehat{\mathbf{k_1}})}{(\alpha^2+|\mathbf{k_1}-\ \boldsymbol{\tau}|^2)(\hbar^2\beta^2+|\mathbf{k_1}-\ \boldsymbol{\tau}|^2)^2(4\hbar^2\gamma^2+|\mathbf{k_1}-\ \boldsymbol{\rho}|^2)(4\hbar^2\gamma^2+|\mathbf{k_1}-\ \boldsymbol{\sigma}|^2)}\end{aligned} \quad (178)$$

Now recalling the definitions of each vector and the expansions of the squares of the differences in each vector, we define the following constants,

$$
\begin{aligned}
C_1 &= \alpha^2 + k_1^2 + \tau^2 - 2k_1\tau cos\theta_{k_1} cos\theta_\tau \ , && (179)\\
C_2 &= -2k_1\tau sin\theta_{k_1} sin\theta_\tau cos\phi_\tau \ , && (180)\\
C_3 &= -2k_1\tau sin\theta_{k_1} sin\theta_\tau sin\phi_\tau \ , && (181)\\
C_4 &= \beta^2 + k_1^2 + \tau^2 - 2k_1\tau cos\theta_{k_1} cos\theta_\tau \ , && (182)\\
C_5 &= \beta^2 + k_1^2 + \rho^2 - 2k_1\rho cos\theta_{k_1} cos\theta_\rho \ , && (183)\\
C_6 &= -2k_1\rho sin\theta_{k_1} sin\theta_\rho cos\phi_\rho \ , && (184)\\
C_7 &= -2k_1\rho sin\theta_{k_1} sin\theta_\rho sin\phi_\rho \ , && (185)\\
C_8 &= \beta^2 + k_1^2 + \sigma^2 - 2k_1\sigma cos\theta_{k_1} cos\theta_\sigma \ , && (186)\\
C_9 &= -2k_1\sigma sin\theta_{k_1} sin\theta_\sigma cos\phi_\sigma \ , && (187)\\
C_{10} &= -2k_1\sigma sin\theta_{k_1} sin\theta_\sigma sin\phi_\sigma \ . && (188)
\end{aligned}
$$

With these substitutions, we can isolate the integration over ϕ_{k_1}. By recalling again that

$$
Y_{l,m}(\theta,\phi) = (-1)^{(m+|m|)/2} P_{l,m}(cos\theta) e^{im\phi} \ , \qquad (189)
$$

we factor the associated Legendre polynomials out of the ϕ_{k_1} integral, and write it as

$$
\int_0^{2\pi} d\phi_{k_1} \frac{e^{i(m_1-m)\phi_{k_1}}}{(C_1 + C_2 cos\phi_{k_1} + C_3 sin\phi_{k_1})(C_4 + C_2 cos\phi_{k_1} + C_3 sin\phi_{k_1})^2 (C_5 + C_6 cos\phi_{k_1} + C_7 sin\phi_{k_1})} \times \frac{1}{(C_8 + C_9 cos\phi_{k_1} + C_{10} sin\phi_{k_1})} \qquad (190)
$$

This integral can be integrated by transforming it into a contour integral. The following substitutions are made,

$$
\begin{aligned}
e^{i\phi_{k_1}} &= z \ , && (191)\\
d\phi &= -iz^{-1} dz \ , && (192)\\
cos\phi_{k_1} &= \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z} \ , && (193)\\
sin\phi_{k_1} &= \frac{z-z^{-1}}{2i} == \frac{z^2-1}{2iz} \ . && (194)
\end{aligned}
$$

With these substitutions, we see that as ϕ_{k_1} ranges between 0 and 2π, our new variable of integration traces out the unit circle in the complex plane. Thus the integral is now a contour integral,

$$
-2^5 i \int_C dz \frac{z^4 z^{m_1-m}}{((C_2 - iC_3)z^2 + 2C_1 z + (C_2 + iC_3))((C_2 - iC_3)z^2 + 2C_4 z + (C_2 + iC_3))^2} \times \frac{1}{((C_6 - iC_7)z^2 + 2C_5 z + (C_6 + iC_7))((C_9 - iC_{10})z^2 + 2C_8 z + (C_9 + iC_{10}))} \ . \qquad (195)
$$

The problem is now reduced to finding the poles of this equation, so that it can be evaluated in terms of the sum of the residues inside the unit circle. Taking as an example the first of the four second order polynomials that can be identified in the denominator, the quadratic formula gives the roots,

$$z_{1\pm} \quad = \quad \left\{ \frac{C_1}{\sqrt{C_2^2 + C_3^2}} \mp \sqrt{\frac{C_1^2}{C_2^2 + C_3^2} - 1} \right\} e^{i\phi_\tau} \ , \tag{196}$$

Where the definitions of the C_i have been utilized to reduce this result into the given polar format. To see where these roots lie, we first recognize that

$$\frac{C_1}{\sqrt{C_2^2 + C_3^2}} \quad > \quad 1 \ . \tag{197}$$

This can be proven by using the definitions of the C_i and showing that the numerator is always greater than the denominator. The following chain of inequalities proves the relation, inserting for C_1, C_2 and C_3,

$$\begin{aligned}
C_1^2 - C_2^2 - C_3^2 \quad &= \quad (\alpha^2 + k_1^2 + \tau^2)^2 - 4(k_1^2 + \tau^2)k_1\tau cos\theta_{k_1} cos\theta_\tau + 4k_1^2\tau^2 cos^2\theta_{k_1} cos^2\theta_\tau \\
&\quad -4k_1^2\tau^2 sin^2\theta_{k_1} sin^2\theta_\tau && (198) \\
&> \quad (k_1^2 + \tau^2)^2 - 4(k_1^2 + \tau^2)k_1\tau cos\theta_{k_1} cos\theta_\tau + 4k_1^2\tau^2 cos^2\theta_{k_1} cos^2\theta_\tau \\
&\quad -4k_1^2\tau^2 sin^2\theta_{k_1} sin^2\theta_\tau && (199) \\
&\geq \quad (k_1^2 + \tau^2)^2 - 4(k_1^2 + \tau^2)k_1\tau + 4k_1^2\tau^2 && (200) \\
&= \quad (k_1 - \tau)^4 && (201) \\
&\geq \quad 0 \ , && (202)
\end{aligned}$$

and so giving

$$C_1^2 \quad > \quad C_2^2 + C_3^2 \ . \tag{203}$$

The only real difficulty is in the relationship between equations (199) and (200). This relationship can be proven by showing that the minima of the second line of the chain is given by $\{\theta_{k_1}, \theta_\tau\} = \{0, 0\}$ and $\{\pi, \pi\}$ by differentiating first with respect to θ_{k_1} to find its extrema, and then by θ_τ to find its extrema. Where these coincide, the function overall has either a minimum, maximum, or saddle point. The two equations that must be satisfied are

$$\begin{aligned}
4(k_1^2 + \tau^2)k_1\tau sin\theta_{k_1} cos\theta_\tau - 8k_1^2\tau^2 sin\theta_{k_1} cos\theta_\tau \quad &= \quad 0 \ , && (204) \\
4(k_1^2 + \tau^2)k_1\tau sin\theta_\tau cos\theta_{k_1} - 8k_1^2\tau^2 sin\theta_\tau cos\theta_{k_1} \quad &= \quad 0 \ . && (205)
\end{aligned}$$

The two equations are simultaneously satisfied at the points $\{\theta_{k_1}, \theta_\tau\} = \{0, 0\}, \{\pi/2, \pi/2\}, \{\pi, \pi\}$. Further investigation reveals that the $\{\pi/2, \pi/2\}$ represents a maximum, while the other two represent minima. The exception to this is when $k_1 = \tau$, when the entire line $\theta_{k_1} = \theta_\tau$ is a solution, but one which gives $C_1^2 - C_2^2 - C_3^2 = \alpha^2 > 0$ in any case, holding the relation given by the inequality chain.

The boundaries of course must be checked, and for these the final result of the inequality chain still holds, but line three of the relation does not necessarily hold along the boundaries. We begin by investigating the partial derivatives given above with one or the other θ coordinate set to a constant along the boundary. This again gives three possible solutions: the two end points of the boundary segment, and an intermediate point that is given by one angle set at π or 0 and the other at

$$cos\theta = \pm\frac{k_1^2+\tau^2}{2k_1\tau} , \tag{206}$$

where the positive case corresponds to the two boundaries for which one of the θ coordinates equals zero and the minus where one of the θ coordinates equals π.

This adds the points $\{\theta_{k_1}, \theta_\tau\} = \{0, \pi\}, \{\pi, 0\}$ and the intermediary points to our consideration. For the corner points we find

$$\begin{aligned} C_1^2 - C_2^2 - C_3^2 &= (\alpha^2 + k_1^2 + \tau^2) + 4(k_1^2 + \tau^2)k_1\tau + 4k_1^2\tau^2 & (207)\\ &> (k_1+\tau)^4 & (208)\\ &\geq (k_1-\tau)^4 & (209)\\ &\geq 0 , & (210)\end{aligned}$$

so that the full relation holds. For the four intermediate points we find

$$\begin{aligned} C_1^2 - C_2^2 - C_3^2 &= (\alpha^2 + k_1^2 + \tau^2)^2 - 2(k_1^2+\tau^2)^2 + (k_1^2+\tau^2)^2 & (211)\\ &> 2(k_1^2+\tau^2)^2 - 2(k_1^2+\tau^2)^2 & (212)\\ &= 0 . & (213)\end{aligned}$$

Thus, $C_1^2 - C_2^2 - C_3^2$ is always greater than zero.

With the fact that the leading term of the root in equation (196) is always greater than one, we see that the midpoint between the two roots lie outside the contour. Thus the larger root, z_{1-} lies outside the contour. It turns out that the other root, z_{1+} must always lie inside the contour. To see this is so, we consider the function

$$f(x) = x - \sqrt{x^2-1} , \tag{214}$$

which describes the position of the z_{1-} root and is always less than 1 when $x > 0$, and we have already shown that $x > 0$. All the roots are found in the same way, and so we find that there are four poles inside the contour, three simple and one that is a second order pole. With the roots in hand, the integral can immediately be evaluated using the residue theorem. The other poles are

$$z_{2\pm} = \left\{\frac{C_4}{\sqrt{C_2^2+C_3^2}} \mp \sqrt{\frac{C_4^2}{C_2^2+C_3^2} - 1}\right\} e^{i\phi_\tau} , \tag{215}$$

$$z_{3\pm} = \left\{\frac{C_5}{\sqrt{C_6^2+C_7^2}} \mp \sqrt{\frac{C_5^2}{C_6^2+C_7^2} - 1}\right\} e^{i\phi_\rho} , \tag{216}$$

$$z_{4\pm} = \left\{\frac{C_8}{\sqrt{C_9^2+C_{10}^2}} \mp \sqrt{\frac{C_8^2}{C_9^2+C_{10}^2} - 1}\right\} e^{i\phi_\sigma} . \tag{217}$$

The integral can be rewritten in terms of it's roots,

$$
\begin{aligned}
-2^5 i \int_C F(z) &= \frac{-2^5 i}{(C_2 - iC_3)^3 (C_6 - iC_7)(C_9 - iC_{10})} \\
&\quad \times \int_C \frac{z^4 z^{m_1 - m}}{(z - z_{1+})(z - z_{1-})(z - z_{2+})^2 (z - z_{2-})^2} \\
&\quad \times \frac{1}{(z - z_{3+})(z - z_{3-})(z - z_{4+})(z - z_{4-})} \ , \qquad (218)
\end{aligned}
$$

and it can be evaluated using the residue theorem,

$$
-2^5 i \int_C F(z) = 2^6 \pi \sum (residues) \ . \qquad (219)
$$

Now the residues must be found. The residues from the simple poles are easiest to find, and using the factored form of $F(z)$ we see that they are

$$
\begin{aligned}
(res)_1 &= \left. \frac{z^4 z^{m_1 - m}}{(z - z_{1-})(z - z_{2+})^2 (z - z_{2-})^2 (z - z_{3+})(z - z_{3-})(z - z_{4+})(z - z_{4-})} \right|_{z = z1+} \ , \qquad (220) \\
(res)_3 &= \left. \frac{z^4 z^{m_1 - m}}{(z - z_{1+})(z - z_{1-})(z - z_{2+})^2 (z - z_{2-})^2 (z - z_{3-})(z - z_{4+})(z - z_{4-})} \right|_{z = z3+} \ , \qquad (221) \\
(res)_4 &= \left. \frac{z^4 z^{m_1 - m}}{(z - z_{1+})(z - z_{1-})(z - z_{2+})^2 (z - z_{2-})^2 (z - z_{3+})(z - z_{3-})(z - z_{4-})} \right|_{z = z4+} \ . \qquad (222)
\end{aligned}
$$

The final residue is

$$
\begin{aligned}
(res)_2 &= 2 \frac{d}{dz} ((z - z_{2+})^2 F(z))|_{z_{2+}} \qquad (223) \\
&= 2 (z_{2+})^{3 + m_1 - m} \frac{(4 + m_1 - m) D(z_{2+}) - z_{2+} \ d/dz(D(z))|_{z = z_{2+}}}{D(z_{2+})^2} \ , \qquad (224) \\
D(z) &= (z - z_{1+})(z - z_{1-})(z - z_{2-})^2 (z - z_{3+})(z - z_{3-})(z - z_{4+})(z - z_{4-}) \ , \qquad (225)
\end{aligned}
$$

with the indicated derivative of the denominator being understandably lengthy enough not to merit writing it fully here.

Finally, we have that the ϕ_{k_1} integral is

$$
\frac{2^6 \pi ((res)_1 + (res)_2 + (res)_3 + (res)_4)}{(C_2 - iC_3)^3 (C_6 - iC_7)(C_9 - iC_{10})} \equiv f_{m_1 - m}(k_1, \theta_{k_1}, \mathbf{k_2}, \mathbf{K}, \mathbf{K_0}) \ . \qquad (226)
$$

Where we have defined a shorthand for this piece of the overall integral, and it is important to note the dependence of the integral on $m_1 - m$, which is key in investigating the helicity asymmetry further on.

Going back to the original integral, inserting for the completed ϕ_{k_1} integration, we have

$$16N\sqrt{\frac{2\hbar}{\pi}}\hbar\sqrt{24\pi}\sum_{m}\sum_{l_1+l_2=1}\sum_{m_1,m_2}C^{1,1}_{l_1,m_1,l_2,m_2}\frac{(-1)^{(m+|m|+m_1+|m_1|)/2+l_2}\tau^{l_2}}{\sqrt{(2l_1+1)!(2l_2+1)!}}Y_{l_2,m_2}(\widehat{\boldsymbol{\tau}})$$

$$\times\int dk_1\ \frac{k_1^{l+2}k_1^{l_1}}{(k_1^2+\hbar^2\beta^2)^2(z-T(k_1,k_2,K,P)}$$

$$\times\int sin\theta_{k_1}d\theta_{k_1}d\phi_{k_1}P_{1,|m|}(\theta_{k_1})P_{l_1,|m_1|}(\theta_{k_1})f_{m_1-m}(k_1,\theta_{k_1},\mathbf{k_2},\mathbf{K},\mathbf{K_0}) \qquad (227)$$

$$\equiv\sum_{m}\sum_{l_1+l_2=1}\sum_{m_1,m_2}C^{1,1}_{l_1,m_1,l_2,m_2}Y_{l_2,m_2}(\widehat{\boldsymbol{\tau}})I_{A,l_1,l_2,m,m_1}(\mathbf{k_2},\mathbf{K},\mathbf{K_0})\ , \qquad (228)$$

Where I_{A,l_1,l_2,m,m_1} has been defined as a term dependent on m_1-m, and that is a function of the given coordinates.