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Radiation from an Accelerated Charge and the Principle of Equivalence

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The connection between an accelerated charge and one at rest in a (weak) gravitational field is discussed in accordance with the principle of equivalence. For that purpose, the fields produced by a freely falling charge and a supported one (i.e., at rest in a gravitational field) are transformed to the rest-frame of the observer, who may be similarly supported or freely falling. A nonvanishing energy flux is found only if the charge is freely falling and the observer supported, or *vice versa*. This agrees with previously established results.

INTRODUCTION

The problem of electromagnetic radiation from an accelerated charge has been a matter of controversy for a long time. Even in the simple case of a *uniformly* accelerated charge there are conflicting statements in the literature as to whether radiation is, or is not, emitted. This is despite the fact that an exact solution of Maxwell's equations for the field produced by such a charge was obtained by Born[1] almost sixty years ago. On the basis of this solution Pauli[2] and von Laue[3] claimed that a uniformly accelerated charge does not radiate, while Schott,[4] Milner,[5] Synge,[6] and Fulton and Rohrlich[7] drew the opposite conclusion. This confusion was aggravated by considering the problem in conjunction with the principle of equivalence, according to which a uniformly accelerated object behaves exactly like one at rest in a uniform gravitational field. If, therefore, an accelerated charge does radiate, then so should a charge at rest in a gravitational field. But this immediately raises a further question: Will a supported observer (i.e., one at rest in a gravitational field) and a freely falling one both agree on this phenomenon?

The foregoing question has been dealt with by Rohrlich,[8] who considered a static, homogeneous, gravitational field for which the Riemann–Christoffel tensor vanishes everywhere. He has shown that it is possible in this case to set up a metric involving an arbitrary function which describes phenomena from a supported observer's point of view, and concluded that a freely falling observer would see a supported charge radiating, and that the same is true of a supported observer and a freely falling charge. However, Rohrlich's treatment is not very satisfactory; it is well known that when the Riemann–Christoffel tensor vanishes everywhere, coordinates can be chosen for which the metric is Minkowskian, and thus the function introduced by Rohrlich merely reflects the arbitrariness of the coordinate system. It is more natural to regard a homogeneous gravitational field as a limiting case of a true gravitational field, the limit being that in which the Riemann–Christoffel tensor vanishes. A simple example, and this is the one we shall use, is afforded by the weak Schwarzchild field; the procedure is, in fact, the one which is used to deduce Newton's law of attraction from Schwarzchild's solution.[9] The world line of a stationary object (a charge, or an observer) in such a field is then a natural representation of a supported object, and instantaneous rest-frames can be erected on it by the use of

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[1] M. Born, Ann. Physik, **30**, 1 (1909).
[2] W. Pauli, *Theory of Relativity* (Pergamon Press, Inc., London, 1958).
[3] M. von Laue, *Relativitäts Theorie* (Frederick Vieweg und Sohn, Braunschweig, Germany, 1919), 3rd ed., Vol. 1.
[4] G. A. Schott, *Electromagnetic Radiation* (Cambridge University Press, London, 1912).
[5] S. M. Milner, Phil. Mag. **41**, 405 (1921).
[6] J. L. Synge, *Relativity: The Special Theory* (North-Holland Publ. Co., Amsterdam, 1964), 2nd ed.
[7] T. Fulton and F. Rohrlich, Ann. Phys. **9**, 499 (1960).
[8] F. Rohrlich, Ann. Phys. **22**, 169 (1963).
[9] J. L. Synge, *Relativity: The General Theory* (North-Holland Publ. Co., Amsterdam, 1960).

differential geometry. Although the conclusions we shall arrive at are identical with Rohrlich's, we believe that the method used in this paper conforms better with the spirit of general relativity.

I. INSTANTANEOUS REST-FRAMES FOR A SUPPORTED OBJECT

The world line of a stationary object in the central (Schwarzchild) field of a mass m is easily shown to satisfy the following invariant Frenet-Serret formulae (Ref. 8, pp. 10 and 284).

$$\delta\lambda_{(4)}{}^{i}/\delta s = b\lambda_{(3)}{}^{i}, \qquad (i=1,2,3,4).$$
$$\delta\lambda_{(3)}{}^{i}/\delta s = b\lambda_{(4)}{}^{i},$$
$$\delta\lambda_{(1)}{}^{i}/\delta s = \delta\lambda_{(2)}{}^{i}/\delta s = 0, \tag{1}$$

with

$$b = (Gm/r^2)(1-2Gm/c^2r)^{-1/2},$$

where G is the gravitational constant. $\lambda_{(4)}{}^{i}$ is the (time-like) unit tangent to the world line, $\lambda_{(3)}{}^{i}$, $\lambda_{(1)}{}^{i}$, and $\lambda_{(2)}{}^{i}$ are the first, second, and third (space-like) unit normals, and $\delta/\delta s$ denotes absolute differentiation with respect to the arc length. The scalar b is the first curvature; it is constant, and for a weak field it reduces to $b=Gm/r^2$, which is, of course, the Newtonian acceleration of gravity. This weak field approximation effectively simulates a homogeneous gravitational field. Proceeding on this approximation, we can replace absolute derivatives by ordinary ones and integrate the Eqs. (1). The result is

$$\lambda_{(1)}{}^{i} = (1,0,0,0),$$
$$\lambda_{(2)}{}^{i} = (0,1,0,0),$$
$$\lambda_{(3)}{}^{i} = (0,0,\cosh bs, \sinh bs),$$
$$\lambda_{(4)}{}^{i} = (0,0,\sinh bs, \cosh bs). \tag{2}$$

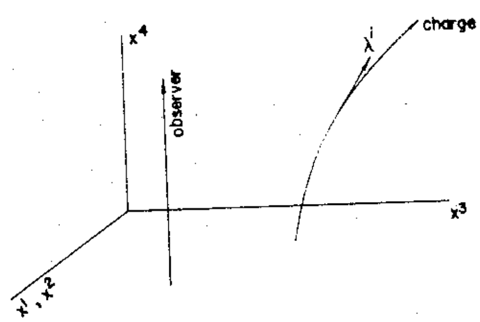

FIG. 1. Supported charge and freely falling observer.

Since $\lambda_{(4)}{}^{i}$ is the unit tangent dx^i/ds, we have

$$x^3 = b^{-1}\cosh bs, \qquad x^4 = b^{-1}\sinh bs, \tag{3}$$

which are the equations of a hyperbola in the (x^3, x^4) plane. Examples of such hyperbolae, together with the orthonormal tetrads $\lambda_{(1)}{}^{i}, \cdots, \lambda_{(4)}{}^{i}$, are shown in Figs. 1 and 2.

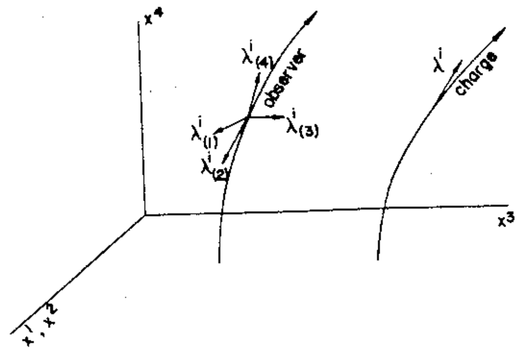

FIG. 2. Supported charge and supported observer.

The $\lambda_{(a)}{}^{i}$'s constitute an instantaneous rectangular frame of reference which is carried along by the object as it follows the world line. For any tensor C_{ij} one can define a set of symbols $C_{(ab)}$ by[9]

$$C_{(ab)} = C_{ij}\lambda_{(a)}{}^{i}\lambda_{(b)}{}^{j}. \tag{4}$$

These symbols are called the components of C_{ij} along the orthonormal tetrad $\lambda_{(a)}{}^{i}$.[10] They are invariants in the tensorial sense, but they depend, of course, on the particular orthonormal tetrad used. The labels (a), (b) may be raised and lowered with the aid of the Minkowskian metric. The procedure of taking the components of a tensor with respect to the orthonormal tetrad on the world line of an observer is equivalent to transforming the tensor to the instantaneous rest-frame of that observer.[9] In what follows, we shall apply this procedure to the electromagnetic tensor F_{ij}.

II. CALCULATION OF THE ELECTROMAGNETIC FIELD AND THE POYNTING VECTOR

To calculate the electromagnetic field produced by a charge we use the well known formula

$$F_{rs} = (\partial\phi_s/\partial x^r) - (\partial\phi_r/\partial x^s), \tag{5}$$

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[10] The name "Vierbein components" is also used.

the four-potential ϕ_r being given by

$$\phi_r = e\left(\frac{A'\lambda_r'}{\lambda_n'(x'^n - x^n)} - \frac{A''\lambda_r''}{\lambda_n''(x''^n - x^n)}\right), \tag{6}$$

where e is the electric charge, x is the event at which ϕ_r is calculated, x' and x'' are the events which the null cone with vertex at x cuts the charge's world line and λ_r', λ_r'' are the respective unit tangents to the world line at those events.[6] The calculation of ϕ_r according to Eq. (6) is found by weighting the contributions from these two events (retarded and advanced) with the two constants A', A'' which are subject to the single condition

$$A' + A'' = 1. \tag{7}$$

We shall now systematically treat the four cases arising from the combination of the two possibilities—supported and freely falling—for the observer and for the charge. In each case we shall state the result of calculating the field according to the formulae (5)–(7). It turns out that this does not depend on any particular choice [subject to (7)] of A' and A''. The electromagnetic tensor F_{rs} is then projected onto the orthonormal tetrad $\lambda_{(a)}{}^r$ on the observer's world line to yield the observed field $F_{(ab)}$. Finally, the $F_{(ab)}$'s are used to construct, in the usual manner, the Poynting vector, the nonvanishing of which we take as the criterion for the existence of radiation measured by the observer at any point on his world line.

A. Freely Falling Observer and Freely Falling Charge

In this case the charge produces the familiar Coulomb field. The orthonormal tetrad on the observer's world line is simply $\lambda_{(a)}{}^i = \delta_a{}^i$ (the Kronecker symbol), so that the field components do not change upon projection onto the $\lambda_{(a)}{}^i$. Now the Poynting vector vanishes for the Coulomb field, and consequently there is no radiation flux.

B. Freely Falling Observer and Supported Charge

The situation is depicted in Fig. 1. The charge follows hyperbolic motion with acceleration b [Eq. (1)] and the field is given by

$$\begin{aligned}
F^{12} &= H_3 = 0,\\
F^{13} &= -H_2 = ebx^1x^4/B^3,\\
F^{23} &= H_1 = ebx^2x^4/B^3,\\
F^{14} &= -E_1 = ebx^1x^3/B^3\\
F^{24} &= -E_2 = ebx^2x^3/B^3,\\
F^{34} &= -E_3 = e(b\sigma + k)/B^3,
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
k &= -(b/2)(\sigma + b^{-2} + \rho^2),\\
\sigma &= (x^3)^2 - (x^4)^2,\\
B^2 &= k^2 - \sigma,\\
\rho^2 &= (x^1 - y^1)^2 + (x^2 - y^2)^2,\\
b^{-2} &= (y^4)^2 - (y^3)^2,
\end{aligned}$$

x and y referring to the positions (in space–time) of observer and charge, respectively. This is Born's solution. As in case a, we have $\lambda_{(a)}{}^i = \delta_a{}^i$, so $F_{(ab)} = F_{ab}$, but now the Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{H}$ is given by

$$\begin{aligned}
S_A &= -(ce^2/4\pi)(b\sigma + k)x^A x^4/B^6, \qquad A = 1, 2\\
S_3 &= (ce^2/4\pi)b^2\rho^2 x^3 x^4/B^6,
\end{aligned} \tag{9}$$

in agreement with previous results.[7] Thus in general the observer encounters radiation. Nevertheless, $\mathbf{S}$ does vanish for certain special events—for example, those with spatial positions directly above or below the charge. This had led Pauli to claim that there is no radiation (Ref. 2, p. 43).

C. Supported Observer and Freely Falling Charge

For simplicity, we shall take the charge to be situated at the origin of space–time. The field it produces is the Coulomb field, and this has to be projected onto the $\lambda_{(a)}{}^i$ of Eq. (2). The resulting Poynting vector is

$$\begin{aligned}
S_A &= -(ce^2/4\pi)b^2x^3x^4x^A/r^6, \qquad A = 1, 2,\\
S_3 &= (ce^2/4\pi)b^2x^3x^4\rho^2/r^6,
\end{aligned} \tag{10}$$

where

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \qquad \rho^2 = (x^1)^2 + (x^2)^2.$$

Since $\mathbf{S} \neq 0$ in general, radiation flux is observed in this case.

D. Supported Observer and Supported Charge

The situation is shown in Fig. 2. The charge again produces the Born field (8), and when this is projected according to Eq. (4) onto the tetrad in Eq. (2), the result is

$$F_{(12)} = F_{(23)} = F_{(31)} = 0,$$

i.e., $\mathbf{H} = 0$,

$$F_{(14)} = E_1 = -(eb^2/B^2)x^1\sigma,$$

$$F_{(24)} = E_2 = -(eb^2/B^3)x^2\sigma,$$

$$F_{(34)} = E_3 = -(e/b^3)(b\sigma + k). \tag{11}$$

Since $\mathbf{H} = 0$ the Poynting vector vanishes and there is no radiation flux.

III. CONCLUSIONS

The results of the previous section, which agree with those of Rohrlich,[2] are indifferent with respect to the observer and the charge. No radiation flux is obtained when both fall freely or when both are supported. The Poynting vector is different from zero only when one of them is supported and the other is in free fall.

These statements are also consistent with the principle of equivalence, according to which they should continue to hold when the word "supported" is changed to read "accelerated." Actually, the principle has already entered the arguments of this paper in the reverse direction when we have passed, in Sec. I, from a supported object to one that follows hyperbolic motion.

It may also be interesting to discuss the foregoing results from a different point of view—that of the photon picture: Is it possible that one of the two observers we have been considering counts a number of photons, while the other, looking at the same charge, does not encounter any of them? In order to answer this question we take the case of the supported charge and the supported observer. Projecting the four-potential of the Born field[1] onto the orthonormal tetrad carried by the supported observer, we find that only the fourth component is different from zero. This means[11] that a radiation detector carried by the observer will not record any transitions in which transverse photons are involved. This is the quantum-electrodynamical explanation for the absence of radiation from a supported charge. It is not enough that photons are there: to be observable, they must be of the transverse kind, and this property—like the nonvanishing of a magnetic field—is not Lorenz invariant.

ACKNOWLEDGMENT

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[11] See, for example, G. Kallen, "Quantenelecktrodynamik" in *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1958), Vol. V, part 1.