Physics 480/581

Problem Session No. 5 Monday, 1 October, 2018

1. Having determined all the equations for a geodesic in the Schwarzschild metric, using our usual orthonormal tetrad, you will know that the last of those 4 equations was the following:

$$\frac{d}{d\tau}\left(\mathcal{H}\frac{dt}{d\tau}\right) + \mathcal{H}'\left(\frac{1}{\mathcal{H}}\frac{dr}{d\tau}\right)\left(\mathcal{H}\frac{dt}{d\tau}\right) = 0 \, .$$

Please show that this can be re-written as a "perfect derivative," i.e., simply the action of $d/d\tau$ on some expression. Therefore you will have discovered a *constant of the motion* for masses moving in this manifold.

2. Consider the second component of the covariant derivative, in the $\hat{\theta}$ direction:

$$\frac{d}{d\tau}\left(r\frac{d\theta}{d\tau}\right) + \frac{\mathcal{H}}{r}\left(\frac{1}{\mathcal{H}}\frac{dr}{d\tau}\right) r\frac{d\theta}{d\tau} - \frac{\cot\theta}{r}\left(r\sin\theta\frac{d\varphi}{d\tau}\right)^2 = 0 \, .$$

It's a second-order differential equation, so one may give it two initial conditions. Show that if one sets $\theta = \pi/2$, the equatorial plane, and also $d\theta/d\tau = 0$ at the initial time, then it will always stay in the equatorial plane.

3. Use the given connections to determine the following two curvature 2-forms, $\underset{\sim}{\Omega}_{\hat{r}\hat{\theta}}$ and $\underset{\sim}{\Omega}_{\hat{\theta}\hat{\varphi}}$, via the generic expression:

$$\underset{\sim}{\Omega}_{\lambda\mu} = d\underset{\sim}{\Gamma}_{\lambda\mu} + \underset{\sim}{\Gamma}_{\lambda\nu} \wedge \underset{\sim}{\Gamma}^{\nu}{}_{\mu} \, .$$

4. Using the Schwarzschild metric, imagine a circle in the equatorial plane, with a constant value of $r > 2M$ and $\theta = \pi/2$, at a fixed time. What is the radius of this circle?