A Coordinate Chart for a Uniformly Accelerated Observer

I. Generalities concerning 4-acceleration

We want to study the trajectory of an accelerated observer, $\mathcal{A}'$, which is modelled as a (parametrized) curve on our manifold with an everywhere timelike tangent vector, $\widetilde{u}$, so that $\mathbf{g}(\widetilde{u},\widetilde{u}) \equiv \widetilde{u}^2 < 0$. We will (usually) look at this worldline via measurements made by a standard inertial observer, $\mathcal{O}$, in flat space. Since this is indeed flat space, within special relativity, her coordinates, which we will call $\{x^\mu \,|\, \mu = 1,2,3,4\} \equiv \{x,y,z,t\}$, can be taken as a coordinate chart for the entire manifold. In these coordinates of course her own worldline is simply the usual $\hat{t}$-axis; i.e., as measured by $\mathcal{O}$, one has simply that $\widetilde{u}_{\mathcal{O}} = \partial_t$. As well, at any fixed moment of her time,i.e., the slice of space-time corresponding to a fixed value of her coordinate, t, is a *3-plane of simultaneity* that is spanned by the vectors $\partial_x, \partial_y, \partial_z$.

For the timelike curve that describes the history of our accelerated observer we choose the affine parameter along that curve to be his proper time, τ, defined so that for displacements along that curve, $(dx)^2 + (dy)^2 + (dz)^2 - (dt)^2 \equiv -(d\tau)^2$, which normalizes its tangent vector, $\widetilde{u}$, so that $\widetilde{u}^2 = -1$. Having this tangent vector, we ask how it varies along the curve; i.e., we define the *acceleration 4-vector*, $\widetilde{a}$, and ask how it varies, as a function of τ:

$$\widetilde{a} \quad \equiv \quad \frac{d}{d\tau}\widetilde{u}\,, \qquad \text{and} \;\; \widetilde{u} = \frac{d}{d\tau} = \frac{dx^\mu}{d\tau}\partial_\mu \,. \tag{1}$$

The derivative along the curve, of the normalization equation for $\widetilde{u}$, tells us that $\widetilde{a}$ is orthogonal to $\widetilde{u}$, and therefore must be spacelike:

$$\frac{d}{d\tau}\{\widetilde{u}^2 = -1\} \quad \Longrightarrow \quad \widetilde{a}\cdot\widetilde{u} = 0 \quad \Longrightarrow \quad \widetilde{a}^2 > 0\,. \tag{2}$$

As the parameter τ increases monotonically along the curve and because we are currently considering the case where his acceleration is positive, the velocity measured by $\mathcal{O}$ will be a monotonically increasing function of τ, which we label as $v(\tau)$. Since our accelerating observer is not an inertial observer, we will associate with it a family of inertial observers $\mathcal{O}''(\tau)$, each of which moves at the constant velocity $v(\tau)$. Therefore when $\mathcal{O}''(\tau)$ observes the accelerated observer at his proper time τ, they measure him to be at rest, although accelerating—just as happens on Earth with a ball thrown into the air, at the top of its trajectory. Since $\widetilde{a}^2 > 0$, we can define a positive scalar A such that $\widetilde{a}^2 = A^2$. Since $\mathcal{O}''(\tau)$ measures $\mathcal{A}'$ to be at rest, he says that $\widetilde{u}_{|_{\mathcal{A}'}} \propto \partial_{t'}$, which gives A the physical interpretation as the magnitude of the 3-acceleration as measured in the inertial frame $\mathcal{O}''(\tau)$; i.e., $A = |\vec{a}|_{\mathcal{O}''}$. Of course, being the square of the 4-acceleration it is an invariant and can be measured in any frame.

II. Motion with (Locally-Measured) Acceleration Uniform

We may now consider the very important special case where the acceleration is uniform, i.e., constant in both magnitude and direction. For simplicity we take the 3-dimensional part of that constant direction to be the direction $\partial_{z''}$. In that case, the equations $\widetilde{a}^2 = A^2$ and $\widetilde{a}\cdot\widetilde{u} = 0$ and $\widetilde{u}^2 = -1$ determine a relation between the coordinates of $\widetilde{a}$ and $\widetilde{u}$:

$$
\begin{aligned}
0 = \widetilde{a}\cdot\widetilde{u} = a^z\,u^z - a^t\,u^t \quad &\Longrightarrow \qquad a^t = \frac{u^z}{u^t}a^z\ ,\\
A^2 = \widetilde{a}^2 = (a^z)^2 - (a^t)^2 = (a^z)^2\left[1 - \left(\frac{u^z}{u^t}\right)^2\right] &= (a^z)^2\frac{(u^t)^2 - (u^z)^2}{(u^t)^2} = \left(\frac{a^z}{u^t}\right)^2 .
\end{aligned}
$$

We intend to consider the case when the accelerating observer has been accelerating from some large negative velocity in the past, comes to rest (instantaneously) at t=0, and then continues with the same constant acceleration into the future of $\mathcal{O}$ with positive velocity. This also arranges that our standard (inertial) observer $\mathcal{O}$ is just $\mathcal{O}$"(0). Therefore we want $u^t > 0$ and also $A > 0$, so that only one of the square roots is appropriate; next, insertion of this into the first line above gives us the second of the following two simple equations:

$$
\frac{du^z}{d\tau} \equiv a^z = A\,u^t\ , \quad \frac{du^t}{d\tau} \equiv a^t = A\,u^z\ . \tag{3}
$$

Inserting $u^z = dz/d\tau$ and $u^t = dt/d\tau$, we may immediately write first integrals of each of these two equations, giving us two constants of integration, b, ℓ:

$$
\frac{d\,z}{d\tau} = A\,t + b\ , \quad \frac{d\,t}{d\tau} = A\,z + \ell\ . \tag{4}
$$

Integration of this pair of equations gives us the (desired) equations for the curve that describes **the trajectory of our uniformly accelerated observer**, parametrized according to his proper time:

$$z = z_0 + \frac{1}{A}(\cosh A\tau - 1) \quad ; \qquad t = t_0 + \frac{1}{A}\sinh A\tau \quad , \tag{5a}$$

We have chosen the zero of τ, i.e., what one might have called τ_0, so that it divides motion inward toward the origin—negative 3-velocity—and motion outward away from the origin—positive 3-velocity, which amounts to symmetrizing the past and the future as measured by the accelerated observer's own clock. At that point, the special observer $\mathcal{O}$ sees the accelerated observer at her coordinates (z_0, t_0), and momentarily at rest, i.e., this choice for τ_0 causes $\mathcal{O} = \mathcal{O}''(0)$, and $(dz/d\tau)_{|\tau=0} = 0$. As well we have chosen the constants z_0 and t_0 as the coordinates at which $\tau = 0$; however, we will now go ahead and take $t_0 \equiv 0$ to simplify the discussion.

We can eliminate the parameter τ from the equations, to discover the "shape" of the curve in spacetime, which gives us the following formula, which is the graph of a hyperbola, symmetric about the z-direction, coming nearest to the origin at $z = z_0, t = t_0 = 0$, and asymptotic to light rays—45°-lines—emitted and absorbed at an event with coordinates $\{z = z_0 - \frac{1}{A}, t = t_0 = 0\}$, which we label as the event $\mathcal{P}$:

$$\left(z - z_0 + \frac{1}{A}\right)^2 - (t)^2 = \frac{1}{A^2} \ , \tag{5b}$$

Notice that we may now differentiate Eqs. (5a) to have explicit presentations of the world velocity and the world acceleration on the proper time:

$$\begin{aligned} u^z &= \sinh(A\tau) \ , \ u^t = \cosh(A\tau) \quad \Longrightarrow \quad v = \tanh(A\tau) \ ; \\ a^z &= A\cosh(A\tau) \ , \ a^t = A\sinh(A\tau) \ . \end{aligned} \tag{5c}$$

To get a "feel" for the magnitude of these quantities, consider the case where we have an observer who accelerates at $A = g = 9.8$ m/sec^2, for ten of his years:

$$A\tau/c = 10.323 \ ; \qquad v = 1 - 2.2 \times 10^{-9} \ ;$$

$$z - z_0 = 1.40 \times 10^{20} \text{ meters } = 14,700 \text{ light-yrs}, \qquad t - t_0 = 14,730 \text{ years.}$$

III. Transformation to an Accelerated Coordinate Chart

We now continue the process of considering measurements made by our accelerated observer, whom we refer to as $\mathcal{A}'$, who is of course not an inertial observer. Nevertheless, at each value of τ, the proper time of $\mathcal{A}'$, our standard observer $\mathcal{O}$ measures him to have a velocity $v = \tanh(A\tau)$, and there always exists an **inertial observer**, $\mathcal{O}''(\tau)$, moving at that speed, who observes $\mathcal{A}'$ to be instantaneously at rest. (We will refer to this inertial observer, $\mathcal{O}''(\tau)$, as the *co-moving observer* at proper time τ.) Therefore, the "(spacelike) 3-planes of simultaneity" that we will need in order to describe $\mathcal{O}'$'s observations will actually be, at least locally, the 3-planes of simultaneity as measured by $\mathcal{O}''(\tau)$.

Then, as is usual, for each value of τ, we set up $\mathcal{O}''(\tau)$'s reference frame so that his origin coincides with that of $\mathcal{O}$; as well they align their measuring rods and clocks. We say that they each have **a choice of frame**, which means that they have chosen a continuously-varying choice of basis for the tangent bundle in some neighborhood of themselves. For $\mathcal{O}$, this choice is simply $\{\partial_x \equiv \widetilde{e}_x,\, \partial_y \equiv \widetilde{e}_y,\, \partial_z \equiv \widetilde{e}_z,\, \partial_t \equiv \widetilde{e}_t\}$, where, of course, $\widetilde{e}_t$ is just the tangent vector to her worldline. Each of the various co-moving observers align their $\widetilde{e}''_t$ axis along their worldline. They also choose their $\widetilde{e}''_x$ and $\widetilde{e}''_y$ axes parallel to those of $\mathcal{O}$. On the other hand, their $\widetilde{e}''_z$ axes will be different from that of $\mathcal{O}$, because it should be perpendicular to the $\widetilde{e}''_t = \widetilde{u}''$ axis.

It is reasonable to choose our accelerating observer as the co-moving observer for the particular time when he is observed by $\mathcal{O}$ to be at rest. This corresponds to choosing t to be zero when τ is zero, i.e., to choose the constant $t_0 = 0$. Since all motion is in the z-direction, it is reasonable to suppose that the accelerated observer can maintain constant the directions of his $\widetilde{e}'_x$ and $\widetilde{e}'_y$; we refer to this as arranging to have **a non-rotating frame**. On the other hand, his timelike basis vector, $\widetilde{e}'_\tau = \widetilde{u}'$ changes as his speed increases; therefore, his $\widetilde{e}'_z$-direction changes as well, being then a function of τ. The basis vector $\widetilde{e}'_z$ must be that direction which, along with $\widetilde{e}'_x, \widetilde{e}'_y$, spans the 3-plane of simultaneity at each value of τ, i.e., the 3-plane perpendicular to $\widetilde{e}_\tau$. Therefore, this vector must be in the same direction as the 4-acceleration, $\widetilde{a}$, since it is perpendicular to $\widetilde{u}$, and our case is the one where $\widetilde{a}$ and $\widetilde{u}$ have no components in the x, y-plane, which is equal to the x', y'-plane. Since both are spacelike, and $|\widetilde{a}| = A$ while $|\widetilde{e}'_z| = +1$, we require that $\widetilde{e}'_z = (1/A)\widetilde{a}$. The trajectory equations, Eqs. (5c), then give us the following relationships:

$$\begin{aligned} \widetilde{e}'_\tau &= \widetilde{u} = \sinh A\tau\, \widetilde{e}_z + \cosh A\tau\, \widetilde{e}_t \ , \\ \widetilde{e}'_z &= \frac{1}{A}\widetilde{a} = \cosh A\tau\, \widetilde{e}_z + \sinh A\tau\, \widetilde{e}_t \quad . \end{aligned} \tag{6}$$

On a 2-dimensional graph, where the x- and y-directions are suppressed as usual, the direction of $\widetilde{e}'_z$ tells us the direction of a "line of simultaneity" for $\mathcal{O}'$, just as the direction of $\widetilde{e}'_\tau = \widetilde{u}$ tells us the direction (at that moment) of the worldline of $\mathcal{O}'$. Continuing to view things in terms of the usual, Minkowski diagram for $\mathcal{O}$, consistently with the statement that the velocity is $v = \tanh(A\tau)$, then the slope of the worldline is just $1/v = \coth A\tau$, while the slope of the line of simultaneity—or the 3-plane of simultaneity if we want to also remember the x and y coordinated—for that τ, is just $v = \tanh A\tau$. These lines of simultaneity, i.e. lines of constant τ, are then simply straight lines with slope $\tanh A\tau$, in $\mathcal{O}$'s Minkowski diagram, all going through the same event $z = z_0 - 1/A, t = 0$, which we have labelled P. Therefore we may write them as $t = (\tanh A\tau)\,(z - z_0 + 1/A)$.

As the accelerated observer moves the lines of simultaneity rotate "upward" as τ increases from zero, and "downward" as it decreases. The light cone at $\mathcal{P}$, two parts of which are the asymptotes of our accelerated observer's hyperbolic motion, separates spacetime into four quadrants. Quadrant **I** is the right-hand spacelike one, which contains the trajectory of our accelerated observer; the future timelike one is labelled **II**, the past timelike one is **IV**, and the left-hand spacelike one is **III**.

To understand these 3-planes somewhat better, and also to determine the coordinates used by the accelerating observer, relative to those used by the stationary observer. As well, from now on let's make some simplifying numerical assignments. We have already taken t_0 to be zero; therefore, let's also choose $z_0 = 1/A$, and then agree to measure everything in units of inverse acceleration, which amounts to also choosing the acceleration, $A = 1$. **Notice** that this chooses the special point P, to be the (common) origin of all our observers, except for the accelerated observer himself, whose origin is now at the point at which he was observed to be at rest, namely at $\mathcal{O}$'s coordinates $(1, 0)$.

We now pick an arbitrary event **in quadrant I**, with coordinates $\{x, y, z, t\}$, as measured by $\mathcal{O}$. It corresponds to some specific, positive proper time, τ, corresponding to the line of simultaneity of $\mathcal{O}'$ that passes through this event, and also consider the co-moving observer, $\mathcal{O}''(\tau)$ for which the accelerated observer is (momentarily) at rest when he observes this event. To understand $\mathcal{O}''(\tau)$ in more detail, we first note that the event at which $\mathcal{O}$ measures our accelerating observer to be instantaneously at rest, i.e., $(1, 0)$ in her coordinates, is in our co-moving observer's past, relative to when he meets with the observer at rest, $\mathcal{O}$. In fact, from the co-moving observer's point of view, his origin, i.e., the event of coordinating his origin with $\mathcal{O}$, is simultaneous with his observation that the accelerating observer is, momentarily, at rest.

To see that this last statement is so, let us note that the co-moving observer always, moves with velocity $v = \tanh(A\tau_0)$, in the positive direction. As well, let's agree to reduce our considerations to one spatial and one temporal dimension, momentarily ignoring the x- and y-directions, in which nothing interesting is happening. Therefore, his worldline, in $\mathcal{O}$'s coordinates, is given by $t = z/v = \coth(A\tau_0)\, z$, while his line of simultaneity for the origin consists of all those points along the line $t = v\, z = \tanh(A\tau_0)\, z$. However, when the accelerating observer has velocity v, and therefore $\tau = \tau_0$, his coordinates are $z = \cosh(A\tau_0)$ and $t = \sinh(A\tau_0)$, which do indeed lie on the line of simultaneity just described above. We can then ask, further, what are the co-moving observer's measurements for the accelerated observer at that time when he is at rest. We have just learned above that the time in question is $t'' = 0$. To determine the distance, we have several possible approaches.

i.) We could notice that one often uses calibration hyperbolae to determine measurements on different worldlines: The relativistic interval is invariant with respect to different observers. Therefore, in particular, the hyperbola $z^2 - t^2 = 1$ is the same as the hyperbola $(z'')^2 - (t'')^2 = 1$. This, however, is the hyperbola on which our accelerating observer is travelling; moreover, when $\mathcal{O}$ observes him to be at rest, he is at $z = 1$, consistent with that hyperbola. Therefore, when $\mathcal{O}''(\tau_0)$ observes him at $t'' = 0$, at rest, he will also be at $z'' = 1$.

ii.) We could use the Lorentz transformation that converts measurements of $\mathcal{O}$ to those of $\mathcal{O}''(\tau_0)$:

$$\begin{pmatrix} z'' \\ t'' \end{pmatrix} = \begin{pmatrix} \cosh(A\tau_0) & -\sinh(A\tau_0) \\ -\sinh(A\tau_0) & \cosh(A\tau_0) \end{pmatrix} \begin{pmatrix} \cosh(A\tau_0) \\ \sinh(A\tau_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{7}$$

iii.) We could even have argued that, **by symmetry**, the accelerated observer's hyperbola is such that it will look the same to any inertial observer, and that it will always be a distance $1/A$ away from the origin at the moment is at rest.

The figure just below is quite useful to help visualize some of this. The (yellow) line through the origin at 45° is a fiduciary light ray, just to help us keep track of slopes; however it is also the asymptote to our accelerated observer's worldline, which is the (thick, red) hyperbola leaving $z = 1$ at time $t = 0$. The uppermost (black) straight line through the origin is the worldline for this particular co-moving observer that we have been discussing, while the (black) line parallel to that, but lower down so that it is tangent to the accelerated worldline, is some observation station of his, at a distance $z'' = 1$ away. That station makes measurements and notices that the accelerated observer passes here, instantaneously at rest, simultaneously with the coincidence of the origins of his frame and that of $\mathcal{O}$. Lastly, the other two straight lines are lines of simultaneity for our particular co-moving observer. The (blue) uppermost one goes through the origin and also the point where the co-moving observer sees the accelerated observer at rest. The (green) lower one goes through the event when the resting observer sees the accelerated one at rest, showing the intersection with $\mathcal{O}$'s worldline, in her past.

Since the points involved were arbitrary, this tells us that every line of simultaneity for our accelerating observer passes through the (common) origin of all our inertial observers! This is indeed a very important observation. [Technically a complete calculation would have shown that all the accelerating observer's lines of simultaneity go through the point with coordinates $(z_0 - 1/A, 0)$, as also do its asymptotes; however, we have normalized this point to the origin by choice of z_0 and A.]

We now compare in detail the coordinates of an arbitrary event, at, say (z_1, t_1) as measured by $\mathcal{O}$, with the coordinates as measured by $\mathcal{A}'$. Of course, those coordinates should actually be the coordinates measured by the appropriate inertial observer that measures the accelerating observer to be at rest, except that those coordinates should be translated by the appropriate amount for the relationship between those two observers. At a proper time τ, the relation between measurements made by $\mathcal{A}'$ and $\mathcal{O}''(\tau)$ is given by

$$z' = z'' - 1 \ , \quad t' = t'' + \tau = \tau \ ,$$

where the last equality comes because, as we have seen, t'' is always zero, i.e., $\mathcal{O}''(\tau)$ always observes the accelerating observer simultaneously with his meeting with the observer at rest, $\mathcal{O}$. On the other hand, the relation between the coordinates of $\mathcal{O}$ and those of $\mathcal{O}''(\tau)$ is given by the appropriate Lorentz transformation:

$$\begin{aligned} z &= z'' \cosh(\tau) + t'' \sinh(\tau) = (z' + 1) \cosh(\tau) \ , \\ t &= t'' \cosh(\tau) + z'' \sinh(\tau) = (z' + 1) \sinh(\tau) \ . \end{aligned} \tag{8}$$

IV. The Neighborhood Where This Transformation is Valid

Before studying in detail the geometry described by these coordinates, we first consider over what neighborhood of spacetime they are valid. It is reasonable that they are not valid for the entirety of space-time, since there are points in spacetime that

(a) can never send information to $\mathcal{A}'$,

(b) some which can never receive information from $\mathcal{A}'$, and

(c) some that fit both of these descriptions. Mathematically, the difficulties with the transformation are of two sorts.

a. Firstly, since all the lines of simultaneity meet at the point $\mathcal{P}$, there is a singularity at that point in the determination of a value of τ for that event.

b. Secondly, for events in quadrants **II** and **III** there are no lines of simultaneity that pass through them; i.e., there are no solutions to the equation that determines the value of τ.

c. Thirdly, for events in quadrant **IV**, as the value of t increases the associated value of τ would decrease, so that the normal ordering given by the progression of time is reversed. This suggests that we have already passed into an unacceptable region. Only **I** is correctly treated by these coordinates.

Note that an equivalent, more elementary approach to the problem is obtained by considering the light ray sent out by $\mathcal{O}$ from her origin. As that light ray is asymptotic to our accelerated observer's worldline, it will intersect that worldline only after infinite proper time. Therefore, no information acquired by $\mathcal{O}$ at any positive times can ever be communicated to the accelerated observer. There are large parts of the spacetime as viewed by $\mathcal{O}$ that he can never observe. We can refer to this particular light ray as **a horizon for our observer**, since it has the property that no light rays can ever pass through it and be received by our accelerated observer.

An approach toward a more general statement, concerning the range of validity of the coordinates of an arbitrarily accelerated observer would lead us to some sort of an approximate statement, at least, that they may not be valid over a neighborhood larger than "of size $1/A$." An interesting additional test one could make would be to consider an observer who moves always at constant velocity **except** for some small time period during which he accelerates from one constant velocity to another. The two associated inertial coordinate systems are locally well-defined, but overlap at larger distances, of the order of $1/A$, in such a way that events outside some neighborhood have two, differing sets of coordinates. If we were to try to ascribe to our semi-accelerated observer some single set of coordinates that covered all of spacetime, it is clear that we would be led into contradictions.

V. Metric Associated with the Accelerated Observer's Coordinate Chart

We now re-write the z', t'-part of Eqs. (8), setting of course $t_0 = 0$ and taking this as an opportunity to define a new coordinate $\xi \equiv z' + 1/A$, but resurrecting generic values for z_0 and A:

$$\begin{aligned} z - z_0 + \frac{1}{A} &= \left(z' + \frac{1}{A}\right) \cosh A\tau \equiv \xi \cosh A\tau \;, \\ t &= \left(z' + \frac{1}{A}\right) \sinh A\tau \equiv \xi \sinh A\tau \,. \end{aligned} \tag{9}$$

The coordinate ξ is simply the one that has the point $\mathcal{P}$ as the origin of its coordinate chart; this causes the accelerated observer to have coordinates $\{\xi = 1/A, \tau = 0\}$ for the event where $\mathcal{O}$ sees it to be momentarily at rest.

Differentiating the above equations, we have the metric in either chart:

$$\mathbf{g} = dx^2 + dy^2 + dz^2 - dt^2 = dx'^2 + dy'^2 + d\xi^2 - A^2\xi^2\, d\tau^2 \;. \tag{10}$$

The coordinate chart is **singular** where $\xi = 0$, i.e., at the point $\mathcal{P}$, which is $1/A$ away from our accelerated observer, just as should, perhaps, have BEEN expected from the discussion above. A second interesting thing to notice, however, is that this metric is exactly the usual Minkowski metric at the point where $\xi = 1/A$, i.e., at the location of our accelerated observer as he measures it; i.e., this coordinate chart is an “equivalence-principle” chart in that its metric is equivalent to that of special relativity in a very near neighborhood of the observer. (The smallness of the neighborhood is of course determined by the accuracy to which one insists that the local metric give the same values for measurements as the special-relativistic metric would give.)

A reasonable (co-tangent) basis for this metric is surely

$$\underset{\sim}{\omega}^x = dx \;, \quad \underset{\sim}{\omega}^y = dy \;, \quad \underset{\sim}{\omega}^z = d\xi \;, \quad \underset{\sim}{\omega}^\tau = A\xi\, d\tau \;, \tag{11a}$$

which gives us a single non-zero connection 1-form, and—as must be so—zero curvature 2-forms:

$$\underset{\sim}{\Gamma}_{z\tau} = \frac{1}{\xi}\underset{\sim}{\omega}^\tau = A\, d\tau \;. \tag{11b}$$

However, we now want to see “what the world looks like” in the accelerating coordinates. Recall that he can really only observe events in quadrant **I**. We may immediately solve the equations to give

$$\xi = \sqrt{z^2 - t^2}\ , \quad A\tau = \tanh^{-1}(t/z)\ . \tag{12}$$

Let’s make a small map of various curves that he may see, on the figure just below, which we now try to explain, noting that we are only showing his observations for $\tau \geq 0$.

His own trajectory is of course just the straight, upward line $\xi = 1/A$. It is useful to plot the paths of light rays headed toward positive, or negative, values of z. Let us first consider those which are sent out toward increasing values of z from an event at $\tau = 0$, which is the same as $t = 0$, and some value a for $\mathcal{O}$’s coordinate z. They correspond to curves $t = z - a$, and therefore satisfy, in our accelerated coordinates the equation $\xi\, e^{-A\tau} = a$. These are curves which indeed begin at $\xi = a$ and $\tau = 0$, headed toward larger values of ξ. However, the curve has the form $A\tau = \log(\xi/a)$, i.e., they are logarithmic in nature, so that they bend over and rise quite slowly. In particular, as a approaches the origin, i.e., approaches zero, then the curve "hugs" the τ-axis longer and longer before it goes on out toward infinity. Notice that when a actually has the value zero, then the only solution to the equation is that ξ must be 0, with τ having any value it wants. This of course says that the particular light ray which is the asymptote to $\mathcal{O}'$'s worldline is mapped into the τ-axis itself.

Now consider paths of light rays headed toward negative values of z. Then the equation is $\xi\, e^{+A\tau} = a$. Again, of course, the paths begin at $\xi = a$, but they now bend toward smaller values of ξ, but also upward, so that they lay closer and closer to the τ-axis, although never touching it, as values of a decrease. Next we may consider stationary observers in $\mathcal{O}$'s frame, at $z = z_0 > 0$, which he may pass as he goes outward. Let them also be stationary at $z = a$; then in his coordinates their path corresponds to $A\tau = \mathrm{Sech}^{-1}(\xi/a)$. These now begin headed vertically, at $\xi = a$ and $\tau = 0$, but quickly turn over and lay down, and, again, begin to "hug" the τ-axis. Now, to complete our picture, we consider some other objects, moving along timelike worldlines with constant velocity v and passing through $z = a$ at $\tau = 0$. Their worldlines would have the form $vt = z - a$. In his coordinates these take the form $\xi = a/[\cosh(A\tau) - v\sinh(A\tau)]$. For $a > 1$, the graph is moving forward at $\tau = 0$, but after some time bends upward and begins to move backward, eventually pressing up against the τ-axis, as usual. Physically this correspond to the accelerated observer coming up to the speed v, and then passing onward to a larger speed. The graph below displays an example of each type of path described above.

We see that the light rays headed toward negative values of z appear never to arrive, but simply take more and more proper time, τ, to asymptotically approach $\xi = 0$. The fact that each of these light rays fails to cross the vertical τ-axis that corresponds to $\xi = 0$ is of course related to the fact that we have a coordinate singularity at $\xi = 0$. It is however even more intimately related to the fact, observed earlier, that this axis is the path of a light ray (emitted at $z = 0$ and $t = 0$) that forms a horizon for our accelerated observer's spacetime. You might recall that this is indeed the same description that is usually given for the behavior of light rays approaching a black hole, as given by some observer who stays carefully far away from that same black hole: the light rays appear to never reach the horizon, but move slowly and more slowly the closer they approach. This horizon, of course, is caused by the behavior of our observer, i.e., he is accelerating; there is, in fact, perfectly ordinary, flat space beyond the horizon, as we know from $\mathcal{O}$'s measurements, which our accelerating observer can of course never receive.

We may also use this entree to consider briefly the notion of "analytic extension" of the metric in question: our metric in coordinates $\{x', y', \xi, \tau\}$. We see that the quadrant of $\mathcal{O}$'s spacetime that we referred to as **quadrant I** has been mapped to the entire $\xi \geq 0$ half-plane. On the other hand, if we simply begin with this metric, and ask about local measurements made by some other observers, we may ask for a coordinate system that might be used, for instance, by observers on one of the paths above, that had the form $A\tau = \mathrm{Sech}^{-1}(\xi/a)$. Since these observers are actually at rest in $\mathcal{O}$'s frame, they would actually reach $\xi = 0$ in some finite amount of their own proper time. Next they would cross over that line and pass out of the realm of observation of our accelerated observer. Nonetheless, we would be able to use them to determine an extension of the $\{\xi, \tau\}$ coordinate system in which their trajectory continues. We can invent a similar set of coordinates, which we refer to as $\{\xi_2, \tau_2\}$, which will continue that path in a smooth way, which are related to the original Minkowski set of coordinates as follows:

$$z = \xi_2 \sinh(A\tau_2) \ , \quad t = \xi_2 \cosh(A\tau_2) \ , \qquad t^2 - z^2 = (\xi_2)^2 \ . \tag{13}$$

This set of coordinates covers **quadrant II** nicely. Notice, however, that the quantity referred to as τ_2 is now a coordinate for spacelike directions, rather than timelike directions as it was in quadrant **I**. If we want to visualize directly how the two sets of coordinates are attached to one another, we need to introduce an intermediate, interpolating quantity. A particular choice of coordinate that does this is to replace ξ, and ξ_2 by R, where

$$\tau = \tau_2 \ , \quad 2R - 1 \equiv z^2 - t^2 = \begin{cases} +\xi^2 \ , & \textbf{quadrant I} \\ -\xi^2 \ , & \textbf{quadrant II} \end{cases} \ ,$$

$$\mathbf{g} = ds^2 = \frac{dR^2}{2R-1} + dy^2 + dz^2 - (2R-1)\, d\tau^2 \ . \tag{14}$$

Since ξ has value 0 on the boundary between the two quadrants, the coordinate R is smoothly defined at that point, having value $R = 1/2$, and creates a metric with many similarities to the standard Schwarzschild metric for a symmetric gravitational system. Notice that which coordinate is timelike and which is spacelike changes when R changes from values greater than 1/2 to those less than that. It is of course true that there is a coordinate singularity at exactly the point $R = 1/2$. There are more complicated choices of coordinates that would remove this problem as well; however, we will discuss these *Kruskal coordinates* when we are discussing the physical problem of spherical symmetry.

We may also consider the original definitions relating $\{z, t\}$ and $\{\xi, \tau\}$, but now consider all values of ξ to be negative, which will give us a map for **quadrant III**, which we could consider as "an analogous presentation for an observer accelerating towards the negative z-direction." Finally, we could use the one referred to above for quadrant **II**, but, once again, allow only negative values for ξ_2, which would give us a map for **quadrant IV**. The metric in terms of R above is valid in all 4 of these quadrants, except, of course, at its coordinate singularities when R has exactly the value $+1/2$.

VI. Rindler Coordinates for our observer's Neighborhood

The coordinates above are useful in that they offer quite a good simulation of the Schwarzschild metric in its properties near the horizon, a place that is troublesome to visualize. As well they allow us to see how we may create a singularity by choice of coordinates, when there truly is no singularity there; i.e., we have created a coordinate singularity. On the other hand, the fact that light rays no longer follow anything like a 45° path is troublesome. Therefore, another popular choice of coordinates is called "Rindler coordinates." These are used by Carroll, for instance, in his last chapter, where he is discussing the radiative decay of black holes, i.e., "Hawking radiation." Therefore, let us also introduce these here, since they are fairly straightforward.

We recall that the coo