Physics 570

Properties of Curvature, for the Levi-Civitá Connection

I. Reminders of the Connection 1-forms and the Curvature 2-forms:

We first recall the first and second structure equations of Cartan, for the important case where general relativity lies, namely with zero torsion, and a metric connection, i.e., a connection where the covariant derivative of the metric vanishes, which is often referred to as the Levi-Civitá connection:

$$
\begin{aligned}
&\textbf{First Structure Equations:}\;\; d\underset{\sim}{\omega}^\lambda - \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\Gamma}^\lambda{}_\mu = 0\ , \\
&\textbf{Second Structure Equations:}\;\; \underset{\sim}{\Omega}^\lambda{}_\mu \;= d\,\underset{\sim}{\Gamma}^\lambda{}_\mu + \underset{\sim}{\Gamma}^\lambda{}_\rho \wedge \underset{\sim}{\Gamma}^\rho{}_\mu\ .
\end{aligned}
\tag{1.1}
$$

We then also recall that the components of the Levi-Civitá connection are given in terms of the ordinary derivatives of the components of the metric tensor and the commutation coefficients of the choice of basis one has made, to determine those components:

$$
\begin{aligned}
&d\underset{\sim}{\omega}^\mu \equiv -\tfrac{1}{2} C_{\nu\lambda}{}^\mu\, \underset{\sim}{\omega}^\nu \wedge \underset{\sim}{\omega}^\lambda\ ; \quad [\widetilde{e}_i, \widetilde{e}_j] \equiv C_{ij}{}^k\, \widetilde{e}_k \quad \text{where}\;\; C_{ij}{}^k = -C_{ji}{}^k\ ; \\
&\underset{\sim}{\Gamma}_{\mu\nu} \equiv g_{\mu\eta}\, \underset{\sim}{\Gamma}^\eta{}_\nu = \Gamma_{\mu\nu\lambda}\, \underset{\sim}{\omega}^\lambda = \tfrac{1}{2}\Big\{(-g_{\nu\lambda\,,\mu} + g_{\mu\nu\,,\lambda} + g_{\mu\lambda\,,\nu}) + (C_{\lambda\nu\mu} + C_{\mu\nu\lambda} + C_{\mu\lambda\nu})\Big\}\, \underset{\sim}{\omega}^\lambda\ ,
\end{aligned}
\tag{1.2}
$$

We remember that if the metric coefficients are constant, then these connection 1-forms are skew-symmetric in their two indices, i.e., $\underset{\sim}{\Gamma}_{\mu\nu} = -\underset{\sim}{\Gamma}_{\nu\mu}$, while if the basis 1-forms commute then the connection 1-forms are given by the Christöffel symbols, which are symmetric in their last two indices, i.e., such that $\left\{ {\mu \atop \nu\lambda} \right\} = \left\{ {\mu \atop \lambda\nu} \right\}$.

The components of the Riemann tensor associated with this connection are then given by

$$
g_{\lambda\rho}\underset{\sim}{\Omega}^\rho{}_\lambda = \underset{\sim}{\Omega}_{\lambda\mu} = \tfrac{1}{2} R_{\lambda\mu\eta\nu}\underset{\sim}{\omega}^\eta \wedge \underset{\sim}{\omega}^\nu\ , \tag{1.3}
$$

where this $\left\{ {0 \atop 4} \right\}$ tensor has an amazing number of symmetries, so that it has only 20 independent components, instead of the 4^4 that one might expect. It is skew-symmetric under interchange of the last two indices, or under the interchange of the first two indices, and is also symmetric under the interchange of those two pairs, as well as satisfying the first Bianchi identity, which is the second line below:

$$\begin{gathered} R_{\mu\nu\lambda\eta} = \; - R_{\mu\nu\eta\lambda} = -R_{\nu\mu\lambda\eta} = +R_{\lambda\eta\mu\nu} \; , \\ R_{\mu\nu\lambda\eta} + R_{\mu\lambda\eta\nu} + R_{\mu\eta\nu\lambda} = 0 \; . \end{gathered} \tag{1.4}$$

From the point of view of transformations of coordinates, or from the point of view of physical meaning, it is useful to divide the curvature tensor into 3 separate parts. Under the usual ideas of group representations, it is always useful to study separately any traces of tensors that exist. Therefore, the first objects to subdivide away from the general form of the curvature tensor are any non-zero traces. As it is skew-symmetric on its 1st and 2nd indices, and also on its 3rd and 4th indices, it is clear that a trace on either of those pairs simply gives zero. Therefore, there is only one non-zero, 2-index trace, usually referred to as the Ricci tensor (although it should be noted that there are two different signs for the Ricci tensor in the literature, depending on whether one sums on the 1st and 3rd index, or on the 1st and 4th one):

$$\mathcal{R}_{\mu\nu} \equiv g^{\lambda\eta} R_{\lambda\mu\eta\nu} = R^{\eta}{}_{\mu\eta\nu} = g^{\lambda\eta} R_{\eta\nu\lambda\mu} = \mathcal{R}_{\nu\mu} \; . \tag{1.5}$$

One can see, in the above equalities, that the Ricci tensor is symmetric in its two indices. However, it also has a non-zero trace, the Ricci scalar:

$$\mathcal{R} \equiv g^{\mu\nu} \mathcal{R}_{\mu\nu} = \mathcal{R}^{\mu}{}_{\nu} \; . \tag{1.6}$$

Therefore, from a group-theoretical point of view, one should take as separate objects the Ricci tensor without its trace, and then, separately, the trace. This, however, is not the best physical way to perform this separation; instead, it is usual to define the so-called Einstein tensor:

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \tfrac{1}{2} \mathcal{R} g_{\mu\nu} \; , \tag{1.7}$$

which is not traceless, but has the very important physical property that it is divergenceless, i.e., we will eventually show that

$$\mathcal{G}^{\mu}{}_{\nu;\mu} = 0 \; . \tag{1.8}$$

The reason for the desire for a divergenceless form of the Ricci tensor is that Einstein's original set of equations will say that we want a form of the Ricci tensor to be proportional to the tensor that describes the **local** matter and energy density of our spacetime. This tensor is required to be divergenceless on grounds that generalize our usual notions of the conservation of energy and momentum, i.e., of 4-momentum; therefore, we will need a divergenceless form of the Ricci tensor to arrange for that proportionality. We will eventually show that the correct form of Einstein's equations say that at every point in spacetime there is an equality between these two tensors:

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = \mathcal{G}_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \ , \tag{1.9}$$

where Λ is the "infamous" cosmological constant, and both sides are in fact divergenceless. (Once again it should be noted that the sign of Λ is sometimes taken as the opposite of this one; however, this is by far the more common one "these days.")
This is of course the source of statements like "the distribution of matter is a cause of the gravitational field." Notice in particular that when one is in a region without energy or matter— so that $T_{\mu\nu}$ is zero— then the Ricci tensor is required to vanish, or to be proportional to the metric via the cosmological constant, if one assumes it is different from zero.

As the Ricci tensor can be viewed as a symmetric, 4×4 matrix, it has 10 independent components; however, the Riemann tensor has 20. Therefore, there are still remaining some 10 additional components. Following Herman Weyl, we will pull those components out into a separate tensor, which is referred to either as the Weyl tensor or the conformal tensor:

$$\begin{aligned} C_{\mu\nu\lambda\eta} &= R_{\mu\nu\lambda\eta} - \tfrac{1}{2}\left(g_{\mu\lambda}\mathcal{R}_{\nu\eta} - g_{\mu\eta}\mathcal{R}_{\nu\lambda} + g_{\nu\eta}\mathcal{R}_{\mu\lambda} - g_{\nu\lambda}\mathcal{R}_{\mu\eta}\right) + \tfrac{1}{6}\left(g_{\mu\lambda}g_{\nu\eta} - g_{\mu\eta}g_{\nu\lambda}\right)\mathcal{R} \\ &= R_{\mu\nu\lambda\eta} - \tfrac{1}{2}\left(g_{\mu[\lambda}\mathcal{R}_{\nu\eta]} - g_{\nu[\lambda}\mathcal{R}_{\mu\eta]}\right) + \tfrac{1}{6}g_{\mu[\lambda}g_{\nu\eta]}\mathcal{R} \ . \end{aligned} \tag{1.10}$$

One can easily notice that the Weyl tensor has the same set of symmetries as does the Riemann tensor. However, in addition, the various extra terms have had their numerical coefficients chosen just so that it has only zero traces. We calculate the trace that gave the Ricci tensor if we had worked with the full Riemann tensor, to show that it is zero:

$$\begin{aligned} g^{\mu\lambda}C_{\mu\nu\lambda\eta} &= g^{\mu\lambda}R_{\mu\nu\lambda\eta} - \tfrac{1}{2}\left(4\mathcal{R}_{\nu\eta} - \delta^{\lambda}_{\eta}\mathcal{R}_{\nu\lambda} + g_{\nu\eta}\mathcal{R} - \delta^{\mu}_{\nu}\mathcal{R}_{\mu\eta}\right) + \tfrac{1}{6}\left(4g_{\nu\eta} - \delta^{\lambda}_{\eta}g_{\nu\lambda}\right)\mathcal{R} \\ &= \mathcal{R}_{\nu\eta} - \tfrac{1}{2}(2\mathcal{R}_{\nu\eta} + g_{\nu\eta}\mathcal{R}) + \tfrac{1}{6}(3g_{\nu\eta})\mathcal{R} = 0 \ . \end{aligned} \tag{1.11}$$

Since this equation amounts to the vanishing of 10 components of the Weyl tensor, it follows that it contains $20 - 10 = 10$ independent components. These are then the components of the curvature that are, at least locally, independent of the density of matter and energy, and can be considered as that part of the curvature caused by the effects of curvature elsewhere, or, if you prefer, by the gravitational field elsewhere.

There is, however, a way to further subdivide the 10 (real-valued) components of the Weyl tensor into 5 (complex-valued) components, via the Hodge duality operation. We recall that for 2-forms, i.e., objects with two skew-symmetric indices, the Hodge duality operation maps them in the following way, where the Hodge dual is denoted by *:

$$\begin{aligned}
\underset{\approx}{\alpha} &= \frac{1}{2}\alpha_{\mu\nu}\,\underset{\approx}{\omega}^{\mu} \wedge \underset{\approx}{\omega}^{\nu} \quad \Longleftrightarrow \quad {}^*\underset{\approx}{\alpha} = \frac{i}{4}\alpha_{\mu\nu} g^{\mu\eta} g^{\nu\lambda} \eta_{\eta\lambda\rho\sigma}\,\underset{\approx}{\omega}^{\rho} \wedge \underset{\approx}{\omega}^{\sigma} \,, \\
\eta^{\alpha\beta\gamma\delta} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 &\equiv \eta^{\alpha\beta\gamma\delta}\underset{\approx}{\mathbf{V}} = \underset{\approx}{\omega}^{\alpha} \wedge \underset{\approx}{\omega}^{\beta} \wedge \underset{\approx}{\omega}^{\gamma} \wedge \underset{\approx}{\omega}^{\delta} \,, \\
\eta^{\alpha\beta\gamma\delta} &= \frac{1}{m}\epsilon[\alpha\beta\gamma\delta] \,, \qquad \eta_{\alpha\beta\gamma\delta} = -\,m\,\epsilon[\alpha\beta\gamma\delta] \,; \qquad \det(g_{\mu\nu}) = -m^2 \,,
\end{aligned} \tag{1.12}$$

where $\mathbf{V}$ is our choice of volume form, as noted by the equality on its left, and $\epsilon[\alpha\beta\gamma\delta]$ is the Levi-Civitá alternating form, such that $\epsilon[1234] = +1$, so that $\eta^{\alpha\beta\gamma\delta}$ is the Levi-Civitá tensor, with $\eta^{1234} = 1/m$.

One may use the Hodge dual to divide the 6-dimensional space of 2-forms into those that are self-dual and those that are anti-self-dual. In principle this is quite straight-forward, since the Hodge dual is an involution, i.e., a mapping that when performed twice is simply the identity, so that its eigenvalues are just $pm1$. Therefore, one simply determines the dual of each of the 6 basis 2-forms, and the three self-dual 2-forms are then the sum of a choice of 3 two-forms, that do **not** transform into each other under duality, added to their duals. In the same fashion the anti-self-dual ones are those same 3 two-forms with their duals subtracted from them. When this is done the two 3-dimensional cotangent spaces that are so created have their own, completely separate connection 1-forms, which are simply appropriate linear combinations of the original 6 connection 1-forms. Since this provides each one of those 3-spaces with a connection, they then also have their own separate curvatures, which are then referred to as the self-dual or anti-self-dual parts of the Riemann tensor.

To describe this more explicitly, we first recall, using the symbols for any **orthonormal** basis, that we have the follow duals of a basis of 2-forms, and that the dual of the dual is itself again:

$$
*\begin{pmatrix} \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^2 \\ \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^3 \\ \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^1 \end{pmatrix} = -i \begin{pmatrix} \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^4 \\ \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^4 \\ \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^4 \end{pmatrix} , \tag{1.13a}
$$

which allow us to write the following basis sets for self-dual and anti-self-dual 2-forms, where there are exactly three elements in each basis set, so that we use (lower-case) Latin indices, which run from 1 to 3, to label them, noting that $\underset{\sim}{\omega}^i$ is the same as $\underset{\sim}{\omega}_i$, while $\underset{\sim}{\omega}_4 = -\underset{\sim}{\omega}^4$:

$$
\underset{\sim}{E}_{i,\pm} \equiv \tfrac{1}{2}\eta_{ijk}\underset{\sim}{\omega}^j \wedge \underset{\sim}{\omega}^k \mp i\underset{\sim}{\omega}_i \wedge \underset{\sim}{\omega}^4 \ . \tag{1.13b}
$$

It of course simplifies computations to note that the anti-self-dual 2-forms are simply the complex conjugates of the self-dual ones.

Then by taking the exterior derivative of these 2-forms, we determine the corresponding set of connection 1-forms which determine their exterior derivatives, showing that the derivatives of self-dual basis 2-forms involve only the self-dual parts of the connections:

$$
\underset{\sim}{G}^i_{\pm} \equiv \tfrac{1}{2}\eta^{ijk}\underset{\sim}{\Gamma}_{jk} \pm i\underset{\sim}{\Gamma}^i{}_4 \ , \qquad d\underset{\sim}{E}_{i,\pm} = \eta_{ijk}\, \underset{\sim}{G}^j_{\pm} \wedge \underset{\sim}{E}^k_{\pm} \ . \tag{1.14}
$$

Since the two subspaces have each their own connections, those (two triplets of) connection 1-forms then generate their own (two triplets of) curvature 2-forms:

$$
\tfrac{1}{2}\eta^{ijk}\underset{\sim}{\Omega}_{jk} \pm i\,\underset{\sim}{\Omega}^i{}_4 \ \equiv \ \underset{\sim}{\Omega}^i_{\pm} = d\underset{\sim}{G}^i_{\pm} - \tfrac{1}{2}\eta^{ijk}\underset{\sim}{G}_{j,\pm} \wedge \underset{\sim}{G}_{k,\pm} \ . \tag{1.15}
$$

In the case of the conformal curvature this simply separates its 10 components into two sets of 5 components, the one set being the complex conjugate of the other set. On the other hand, the Ricci tensor "straddles" this gap and has elements in both subspaces. Since the self-dual part of the conformal tensor has 5 independent (complex) components, a useful way to present them is via a (complex-valued), traceless 3 matrix, which we will describe below.

However, first it is probably useful to display all of the above separations more explicitly. We do this for the case of our **orthonormal** basis choice, and begin with the details for the Ricci tensor:

$$
\begin{aligned}
\mathcal{R}_{11} &= R_{1212} + R_{3131} - R_{1414} \ , \\
\mathcal{R}_{22} &= R_{1212} + R_{2323} - R_{2424} \ , \\
\mathcal{R}_{33} &= R_{3131} + R_{2323} - R_{3434} \ , \\
\mathcal{R}_{44} &= R_{1414} + R_{2424} + R_{3434} \ , \\
\mathcal{R}_{12} &= -R_{1424} + R_{1323} \ , \qquad \mathcal{R}_{13} = -R_{1434} + R_{1232} \ , \\
\mathcal{R}_{23} &= -R_{2434} + R_{1213} \ , \qquad \mathcal{R}_{14} = R_{3134} + R_{2124} \ , \\
\mathcal{R}_{24} &= R_{1214} + R_{3234} \ , \qquad \mathcal{R}_{34} = R_{2324} + R_{1314} \ ; \\
\mathcal{R} &= 2(R_{1212} + R_{2323} + R_{3131} - R_{1414} - R_{2424} - R_{3434}) \ ,
\end{aligned}
\tag{1.16}
$$

And then also for the full conformal tensor, which we write out in stages, using the definition given at Eqs. (1.10). We first look at the "diagonal" ones, i.e., those with the same "first two" indices as the "second two" indices, noting the equalities among the components that, basically, generate the fact that this tensor has no traces and only 10 independent components:

$$
\begin{aligned}
C_{1212} &= R_{1212} - \tfrac{1}{2}(\mathcal{R}_{11} + \mathcal{R}_{22}) + \tfrac{1}{6}\mathcal{R} \\
&= \tfrac{1}{6}(2R_{1212} - 2R_{3434} - R_{3131} - R_{2323} + R_{1414} + R_{2424}) \ = \ -C_{3434} \ , \\
C_{2323} &= R_{2323} - \tfrac{1}{2}(\mathcal{R}_{22} + \mathcal{R}_{33}) + \tfrac{1}{6}\mathcal{R} \\
&= \tfrac{1}{6}(2R_{2323} - 2R_{1414} - R_{1212} - R_{3131} + R_{2424} + R_{3434}) \ = \ -C_{1414} \ , \\
C_{3131} &= R_{3131} - \tfrac{1}{2}(\mathcal{R}_{11} + \mathcal{R}_{33}) + \tfrac{1}{6}\mathcal{R} \\
&= \tfrac{1}{6}(2R_{3131} - 2R_{2424} - R_{1212} - R_{2323} + R_{1414} + R_{3434}) \ = \ -C_{2424} \ .
\end{aligned}
\tag{1.17a}
$$

We note that the Bianchi identity tells us that the three above add to zero, so that only any **two** of them are independent. As well, of course, note that there are 6 different indexed components involved in just these two independent ones.

Next we look at the ones with all 4 indices different:

$$
C_{1234} = R_{1234} \ , \quad C_{2314} = R_{2314} \ , \quad C_{3124} = R_{3124} \ , \tag{1.17b}
$$

noting, again, that these three also add to zero because of the first Bianchi identity, so that we now have **two more** independent ones. Here, however, we do not have any equalities among differently-indexed components, so while so far we have only **four** independent components, we have now used up 9 indexed components. Lastly we look at those which remain, involving one pair of indices all spatial and the other pair the complementary spatial index with the temporal one, there being **six** independent ones here, among a total of 12 differently-indexed components, since each of these six has an equality with a differently-indexed one:

$$\begin{aligned}
C_{1223} &= R_{1223} + \tfrac{1}{2}\mathcal{R}_{31} = \tfrac{1}{2}(R_{1223} + R_{1443}) = C_{1443} \ , \\
C_{2114} &= R_{2114} + \tfrac{1}{2}\mathcal{R}_{24} = \tfrac{1}{2}(R_{2114} - R_{2334}) = -C_{2334} \ , \\
C_{3112} &= R_{3112} + \tfrac{1}{2}\mathcal{R}_{23} = \tfrac{1}{2}(R_{3112} + R_{2443}) = C_{2443} \ , \\
C_{1224} &= R_{1224} + \tfrac{1}{2}\mathcal{R}_{14} = \tfrac{1}{2}(R_{1224} - R_{1334}) = -C_{1334} \ , \\
C_{2331} &= R_{2331} + \tfrac{1}{2}\mathcal{R}_{12} = \tfrac{1}{2}(R_{2331} + R_{1442}) = C_{1442} \ , \\
C_{3224} &= R_{3224} + \tfrac{1}{2}\mathcal{R}_{34} = \tfrac{1}{2}(R_{3224} - R_{3114}) = -C_{3114} \ .
\end{aligned} \tag{1.17c}$$

It is worth pointing out that all the 6 equalities between pairs of conformal tensor components that are noted above come directly from the fact that the conformal tensor has zero trace. That trace, analogous to the Ricci tensor when taking a trace of the Riemann tensor, has 2 indices—on which it is symmetric—and these 6 identities are exactly the content of the off-diagonal elements of that second-rank tensor, which vanishes. One might as well then also point out that the remaining 4 (diagonal) components of that zero tensor may be solved for the 4 identities appearing in the equality pairs already shown in Eqs. (1.17a). Those other 4 (diagonal) equalities, in general, are given by

$$\begin{aligned}
0 &= C_{2121} + C_{3131} - C_{4141} \ , \quad 0 = C_{1212} + C_{3232} - C_{4242} \ , \\
0 &= C_{1313} + C_{2323} - C_{4343} \ , \quad 0 = C_{1414} + C_{2424} + C_{3434} \ .
\end{aligned} \tag{1.17d}$$

We have now listed above the 21 differently-indexed components of this fourth-rank tensor—as expected since we treat it as skew-symmetric in each of the two pairs of indices and symmetric under their interchange. As well we have noted that only 10 of them are independent, as expected. The next task is to divide them up into the 5 self-dual ones and the 5 anti-self-dual ones. Self-duality here is being declared on the symbols as 2-forms, so that we have

$$\begin{aligned} {}^*\underset{\sim}{C}_{\mu\nu} = \; {}^*\big\{\tfrac{1}{2} C_{\mu\nu\lambda\eta}\, \underset{\sim}{\omega}^\lambda \wedge \underset{\sim}{\omega}^\eta\big\} = i\big\{ &C_{\mu\nu 34}\, \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^2 + C_{\mu\nu 24}\, \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^1 + C_{\mu\nu 14}\, \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^3 \\ &- C_{\mu\nu 23}\, \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^4 - C_{\mu\nu 31}\, , \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^4 - C_{\mu\nu 12}\, \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^4 \big\} \,. \end{aligned} \tag{1.18}$$

Therefore the split into self-dual and anti-self-dual parts is given by the following, where we use $\underset{\sim}{\mathcal{C}}_\pm$ to denote the self-dual and anti-self-dual parts of the conformal tensor:

$$\underset{\sim}{\mathcal{C}}_{\pm\mu\nu} = (C_{\mu\nu 12} \pm i C_{\mu\nu 34}) \underset{\sim}{E}^3_\pm + (C_{\mu\nu 31} \pm i C_{\mu\nu 24}) \underset{\sim}{E}^2_\pm + (C_{\mu\nu 23} \pm i C_{\mu\nu 14}) \underset{\sim}{E}^1_\pm \equiv \mathcal{C}_{i\pm\mu\nu} \underset{\sim}{E}^i_\pm \,. \tag{1.19}$$

Inserting the explicit values for μ and ν—two triplets of allowed values—from Eqs. (1.16) above, we may write these out in terms of two sextuplets of quantities, each of which has a quintuplet of independent values (because of the sum below required by the first Bianchi identity). We name them by $A_\pm$, $B_\pm$, $C_\pm$, these three being created from the six independent components listed above in Eqs. (4.5c) and involving only one repeated index, and then $H_\pm$, $J_\pm$, and $K_\pm$, each of which involve a sum of one component from the list of diagonal ones, in Eqs. (1.17a), and one from the list with all four indices different, as noted in Eqs. (1.17b). As both of these last two sets, that make up the $H_\pm$, $J_\pm$, and $K_\pm$, are subject to a requirement from the Bianchi identity, these three also satisfy a Bianchi identity, leaving us with only two pair of independent objects. These are listed below, hopefully in such a way that one can see all the different possible ways to create equivalent indexed names for them:

$$\begin{aligned} \underset{\sim}{\mathcal{C}}_{\pm 12} &= H_\pm \underset{\sim}{E}^3_\pm + A_\pm \underset{\sim}{E}^2_\pm + B_\pm \underset{\sim}{E}^1_\pm = \pm i\, \underset{\sim}{\mathcal{C}}_{\pm 34} \,, \\ \underset{\sim}{\mathcal{C}}_{\pm 31} &= A_\pm \underset{\sim}{E}^3_\pm + K_\pm \underset{\sim}{E}^2_\pm + C_\pm \underset{\sim}{E}^1_\pm = \pm i\, \underset{\sim}{\mathcal{C}}_{\pm 24} \,, \\ \underset{\sim}{\mathcal{C}}_{\pm 23} &= B_\pm \underset{\sim}{E}^3_\pm + C_\pm \underset{\sim}{E}^2_\pm + J_\pm \underset{\sim}{E}^1_\pm = \pm i\, \underset{\sim}{\mathcal{C}}_{\pm 14} \,, \end{aligned} \tag{1.20}$$

with the various parts defined in detail as follows, where the Bianchi identity, in the form

$$\tfrac{2}{3} R_{ijk4} - \tfrac{1}{3} R_{jki4} - \tfrac{1}{3} R_{kij4} = R_{ijk4} - \tfrac{1}{3}(R_{jki4} + R_{ji4k} + R_{j4ki}) = R_{ijk4} \,, \tag{1.21}$$

is used extensively in order to produce the presentations of $H_\pm$, $J_\pm$, and $K_\pm$ in terms of self-dual (or anti-self-dual) parts:

$$
\begin{aligned}
A_\pm &\equiv [C_{1231}=C_{2443}] \pm i\,[C_{1224}=C_{3134}] = \frac{1}{2}\left[(R_{1231}\pm iR_{1224})-(R_{3424}\mp iR_{3431})\right]\ ,\\
B_\pm &\equiv [C_{1223}=C_{1434}] \pm i\,[C_{1214}=C_{2334}] = \frac{1}{2}\left[(R_{1223}\pm iR_{1214})-(R_{3414}\mp iR_{3423})\right]\ ,\\
C_\pm &\equiv [C_{2331}=C_{1442}] \pm i\,[C_{2324}=C_{3114}] = \frac{1}{2}\left[(R_{3123}\pm iR_{3114})+(R_{1442}\mp iR_{2342})\right]\ ,
\end{aligned}
\tag{1.22}
$$

and the other ones:

$$
\begin{aligned}
H_\pm &\equiv [C_{1212}=-C_{3434}] \pm iC_{1234}\\
&= \tfrac{1}{6}(2R_{1212}-2R_{3434}-R_{3131}-R_{2323}+R_{1414}+R_{2424}) \pm iR_{1234}\\
&= \tfrac{1}{6}\,[2(R_{1212}\pm iR_{1234})-2(R_{3434}\mp iR_{1234})-(R_{3131}\pm iR_{3124})\\
&\quad -(R_{2323}\pm iR_{2314})+(R_{1414}\mp iR_{1423})+(R_{2424}\mp iR_{2431})]\\
J_\pm &\equiv [C_{2323}=-C_{1414}] \pm iC_{2314}\ ,\\
&= \tfrac{1}{6}(2R_{2323}-2R_{1414}-R_{1212}-R_{3131}+R_{2424}+R_{3434}) \pm iR_{2314}\\
&= \tfrac{1}{6}\,[2(R_{2323}\pm iR_{2314})-2(R_{1414}\mp iR_{2314})-(R_{1212}\pm iR_{1234})\\
&\quad -(R_{3131}\pm iR_{3124})+(R_{2424}\mp iR_{3124})+(R_{3434}\mp iR_{3412})]\\
K_\pm &\equiv [C_{3131}=-C_{2424}] \pm iC_{3124}\ ,\\
&= \tfrac{1}{6}(2R_{3131}-2R_{2424}-R_{1212}-R_{2323}+R_{1414}+R_{3434}) \pm iR_{3124}\\
&= \tfrac{1}{6}\,[2(R_{3131}\pm iR_{3124})-2(R_{2424}\mp iR_{2431})-(R_{1212}\pm iR_{1234})\\
&\quad -(R_{2323}\pm iR_{2314})+(R_{1414}\mp iR_{1423})+(R_{3434}\mp iR_{3412})]\\
&\text{and}\quad H_\pm + J_\pm + K_\pm = 0\ .
\end{aligned}
\tag{1.23}
$$

II. Presentations of the various parts of the Riemann tensor

One might wonder how to explicitly express all the values of the entire $\left\{ {0 \atop 4} \right\}$ Riemann tensor. The simplest way is to first note that there are 6 independent values—in our 4 dimensions—for a pair of indices that are skew symmetric. Since this tensor has two such pairs, it is best considered as a 6×6 matrix, which also happens to be symmetric, and to have certain traces which vanish. Notice that a general 6×6 matrix has 36 elements; however, a symmetric one has only 21 independent elements. The reduction down to just 20 will be explained via the vanishing of some traces.

Remembering that we are using an **orthonormal basis set**, we will order the 6 basis 2-forms in the following way, which picks out the temporal indices differently:

$$\{\underset{\approx}{\omega}^1 \wedge \underset{\approx}{\omega}^2 \,,\, \underset{\approx}{\omega}^2 \wedge \underset{\approx}{\omega}^3 \,,\, \underset{\approx}{\omega}^3 \wedge \underset{\approx}{\omega}^1 \,,\, \underset{\approx}{\omega}^3 \wedge \underset{\approx}{\omega}^4 \,,\, \underset{\approx}{\omega}^1 \wedge \underset{\approx}{\omega}^4 \,,\, \underset{\approx}{\omega}^2 \wedge \underset{\approx}{\omega}^4\} \,. \tag{2.0}$$

Examples could be the Minkowski choice, namely $\{\hat{x}\hat{y}, \hat{y}\hat{z}, \hat{z}\hat{x}, \hat{z}\hat{t}, \hat{x}\hat{t}, \hat{y}\hat{t}\}$, appropriate in flat space, or, for instance, the orthonormal version of spherical coordinates along with time, $\{\hat{\theta}\hat{\varphi}, \hat{\varphi}\hat{r}, \hat{r}\hat{\theta}; \hat{r}\hat{t}, \hat{\theta}\hat{t}, \hat{\varphi}\hat{t}\}$. Having made that spatial-temporal separation, we then write out our 6×6 matrix in terms of three 3×3 matrices, where we will need to require that both M and Q are symmetric matrices.

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{Q} \end{pmatrix} \,. \tag{2.1}$$

The matrix $\mathbf{M}$ corresponds, per the list above, to Riemann tensor components without any indices that refer to time, i.e., using just the spatial ones given above in Eq. (2.0), **and** in the order shown there, so that, for instance, $R_{1212} = \mathbf{M}_{11}$, $R_{1231} = \mathbf{M}_{13}$, etc. Those of $\mathbf{Q}$ correspond to those components with two 4-indices, i.e., both row and column indices from the second triad of indices shown in Eq. (2.0), and in the order given there, again, so that, for example $R_{3424} = \mathbf{Q}_{13}$. Then the matrix $\mathbf{N}$ contains those components with only one 4 index, so that, for instance $R_{2334} = \mathbf{N}_{21}$, while $\mathbf{N}_{12} = R_{1214}$.

Since $\mathbf{M}$ and $\mathbf{Q}$ are symmetric matrices, they each have 6 independent components, while there are 9 components for N, which is a total of 21. Lastly, however, we know that the first Bianchi identity reduces the total number down to 20, with its one independent constraint, which is only an independent constraint when all 4 indices are distinct, which tells us that the matrix N must be traceless, so that, actually, $\mathbf{N}$ has only 8 degrees of freedom, reducing the number of independent components of this matrix down to 20, the correct answer.

Since Hodge duality is somewhat important in these discussions, I record the duality relations for our 3×3 matrices:

$$*\mathbf{M} = -i\mathbf{N} \; , \quad *Q = +i\mathbf{N}^T \quad \Longrightarrow \quad *R = -i \begin{pmatrix} \mathbf{N} & -\mathbf{M} \\ \mathbf{Q} & -\mathbf{N}^T \end{pmatrix} \; . \tag{2.2}$$

The curvature tensor contains various separate parts, which are of considerable physical interest, as discussed somewhat in the previous section. Therefore, we want to divide up this format so as to see how they appear, which just uses the explicit decompositions given in the previous section. Some are easily presented, and some not quite so. The reason is that our ordering is arranged so that Hodge duals are performed easily, but note below that the spatial components are determined cyclically, so that $1+1=2$, and $2+1=3$, while $3+1=1$:

$$\begin{gathered} \mathcal{R}_{\hat{t}\hat{t}} = \mathrm{trace}(\mathbf{Q}) \; , \; \mathcal{G}_{\hat{t}\hat{t}} = \mathrm{trace}(\mathbf{M}) \; , \quad \mathcal{R} = 2\,\mathrm{trace}(\mathbf{M}-\mathbf{Q}) \; , \\ \mathcal{R}_{\hat{i}\hat{j}} = -(\mathbf{M}+\mathbf{Q})_{i+1,j+1} + \delta_{ij}\mathrm{trace}(\mathbf{M}) \; , \quad \mathcal{G}_{\hat{i}\hat{j}} = -(\mathbf{M}+\mathbf{Q})_{i+1,j+1} + \delta_{ij}\mathrm{trace}(\mathbf{Q}) \; , \\ \mathcal{R}_{\hat{i}\hat{t}} = \epsilon^{i+1,jk}\mathbf{N}_{jk} = \mathcal{G}_{\hat{i}\hat{t}} \; . \end{gathered} \tag{2.3a}$$

To go on to the conformal tensor, again using equalities from the previous section, we use the mapping of indices inherent in the original ordering of the indices translating Riemann skew-pair indices into these 3×3 indices, as discussed in the paragraph after *Eq.* (2.0), so that, for instance $i,j=2,3$ corresponds to $a=2$, etc., and we also use i,j,k, as well as p,q,r in cyclic order, and we also recall the mapping under Hodge duality:

$$\begin{aligned} C_{ijpq} = \tfrac{1}{2}(\mathbf{M}-\mathbf{Q})_{ab} - \tfrac{1}{6}\delta_{ab}\mathrm{trace}(\mathbf{M}-\mathbf{Q}) = -C_{k4r4} = -i * C_{ijr4} \; , \\ C_{ijr4} = \tfrac{1}{2}(\mathbf{N}+\mathbf{N}^T)_{ar} = C_{k4pq} = i * C_{ijpq} \; ; \end{aligned} \tag{2.3b}$$

Because of the division for 2-forms into self-duality and anti-self-duality, the 10 independent components of the conformal, or Weyl, tensor are best presented via a single traceless, symmetric, complex 3×3 matrix—and therefore with 5 complex components. Using the usual identification between elements of these 3×3 matrices, we can define a matrix $\mathbf{P}$ as follows, which is the desired self-dual part

$$\mathbf{P} \equiv \tfrac{1}{2}(\mathbf{M}-\mathbf{Q}) - \tfrac{1}{6}\mathrm{trace}(\mathbf{M}-\mathbf{Q})\mathbf{I}_3 - \tfrac{i}{2}(\mathbf{N}+\mathbf{N}^T) \; , \tag{2.5}$$

where $\mathbf{I}_3$ is the usual 3×3 identity matrix.

For those cases when the Ricci tensor vanishes, i.e., for vacuum solutions of the Einstein field equations, this equation reduces quite substantially. From Eqs. (2.3) above, one can see that vanishing of the Ricci tensor requires $\mathbf{Q} = -\mathbf{M}$ with both traceless, while $\mathbf{N}$ must be symmetric, so that the equation simplifies to

$$\mathbf{P} = \mathbf{M} - i\mathbf{N} \,, \quad \text{for vacuum solutions.} \tag{2.6}$$

The number of eigenvalues, and their degeneracy, of this matrix, $\mathbf{P}$, determines what is referred to as *the Petrov type*, and is an important way to distinguish various classes of vacuum solutions. As well the quantities trace($\mathbf{P}^2$) and trace($\mathbf{P}^3$) are functional invariants of the manifold.

III. The view of these tensors from the usual null basis

We first recall the usual relation between the orthonormal basis used in the previous sections, and the null basis with our sign conventions:

$$\begin{aligned} \underset{\sim}{\theta}^\alpha &= A^\alpha{}_\mu \, \underset{\sim}{\hat{\omega}}^\nu \,, \\ \begin{pmatrix} \underset{\sim}{\theta}^1 \\ \underset{\sim}{\theta}^2 \\ \underset{\sim}{\theta}^3 \\ \underset{\sim}{\theta}^4 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} +1 & +i & 0 & 0 \\ +1 & -i & 0 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & +1 & +1 \end{pmatrix} \begin{pmatrix} \underset{\sim}{\hat{\omega}}^1 \\ \underset{\sim}{\hat{\omega}}^2 \\ \underset{\sim}{\hat{\omega}}^3 \\ \underset{\sim}{\hat{\omega}}^4 \end{pmatrix} . \end{aligned} \tag{3.1}$$

The matrix A then has determinant $-i$, consistent with the fact that the determinant of $\nu_{\alpha\beta}$ is $+1$ while the determinant of $\eta_{\mu\nu}$ is -1. We could then show that the value of the metric, in this basis is indeed $\nu_{\alpha\beta}$ as stated above, by using the inverse of this matrix A to transform $\boldsymbol{\eta}$ into $\boldsymbol{\nu}$, i.e., the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Longleftarrow \quad \nu_{\alpha\beta} = (A^{-1})^\mu{}_\alpha (A^{-1})^\nu{}_\beta \, \eta_{\mu\nu} \,, \quad A^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -i & +i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} . \tag{3.2}$$

The transformation of the Riemann tensor, or the Weyl tensor, into this basis is then accomplished in the usual way, where I insert an overtilde on the symbols when they are in null basis. This distinction, via some symbolism, will be quite important when we look at individual components:

$$\widetilde{R}_{\alpha\beta\gamma\delta} = (A^{-1})^{\mu}{}_{\alpha}(A^{-1})^{\nu}{}_{\beta}(A^{-1})^{\lambda}{}_{\gamma}(A^{-1})^{\eta}{}_{\delta}\, R_{\mu\nu\lambda\eta} \ , \tag{3.3}$$

and of course the same format for the conformal (or Weyl) tensor.

It is true that these tensors, in this other basis, have the same skew-symmetry, and symmetry properties, as in the other basis, including the first Bianchi identity. However, because of the difference in the form of the metric, various identities look considerably different. Let us describe these for the conformal tensor, which has this additional property of having zero trace. We define a zero tensor via that trace:

$$M_{\beta\delta} = 0 \equiv \nu^{\alpha\gamma} C_{\alpha\beta\gamma\delta} = C_{1\beta 2\delta} + C_{2\beta 1\delta} + C_{3\beta 4\delta} + C_{4\beta 3\delta} \ . \tag{3.4}$$

We may then write out the 10 relations that this includes:

$$\begin{gathered} M_{11} = 0 = 2C_{3141} \ , \quad M_{22} = 0 = 2C_{3242} \ , \quad M_{33} = 0 = 2C_{1323} \ , \quad M_{44} = 0 = 2C_{1424} \ , \\ M_{12} = 0 = C_{2112} + C_{3142} + C_{4132} \ , \quad M_{13} = 0 = C_{2113} + C_{3143} \ , \quad M_{14} = 0 = C_{2114} + C_{4134} \ , \\ M_{23} = 0 = C_{1223} + C_{3243} \ , \quad M_{24} = 0 = C_{1224} + C_{4234} \ , \quad M_{34} = 0 = C_{1324} + C_{2314} + C_{4334} \ . \end{gathered} \tag{3.5}$$

The two 3-term relations in the last two lines above easily imply the relation

$$C_{1212} = C_{3434} \ . \tag{3.6}$$

There is some reason to try to present these in the form of something other than a table like the one above; therefore, we want now to put it together in the form of a symmetric 6×6 matrix, as was done for the Riemann tensor—in orthonormal coordinates—in the preceding section. As before, we choose a basis for this 6×6 matrix, i.e., a basis for 2-forms built on the null basis for 1-forms, in what appears—at first glance—to be the same order as we used for the orthonormal one, namely

$$\{\underset{\sim}{\theta}^1 \wedge \underset{\sim}{\theta}^2, \underset{\sim}{\theta}^2 \wedge \underset{\sim}{\theta}^3, \underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^1, \underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^4, \underset{\sim}{\theta}^1 \wedge \underset{\sim}{\theta}^4, \underset{\sim}{\theta}^2 \wedge \underset{\sim}{\theta}^4\} \ , \tag{3.7}$$

although of course, for instance, $\underset{\sim}{\theta}^2 \wedge \underset{\sim}{\theta}^3$ is much different from $\underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^3$. In fact, we will here present the relation between them in the form of a matrix which relates one set to the other:

$$\begin{pmatrix} \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^2 \\ \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^3 \\ \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^1 \\ \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^4 \\ \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^4 \\ \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2i & 0 & 0 & 0 & 0 & 0 \\ 0 & +i & +i & 0 & -i & +i \\ 0 & -1 & +1 & 0 & -1 & -1 \\ 0 & 0 & 0 & +2 & 0 & 0 \\ 0 & -1 & +1 & 0 & +1 & +1 \\ 0 & -i & -i & 0 & -i & +i \end{pmatrix} \begin{pmatrix} \underset{\sim}{\theta}^1 \wedge \underset{\sim}{\theta}^2 \\ \underset{\sim}{\theta}^2 \wedge \underset{\sim}{\theta}^3 \\ \underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^1 \\ \underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^4 \\ \underset{\sim}{\theta}^1 \wedge \underset{\sim}{\theta}^4 \\ \underset{\sim}{\theta}^2 \wedge \underset{\sim}{\theta}^4 \end{pmatrix} , \tag{3.8}$$

where we will refer to the square matrix above as H.

If we then write out the conformal, or Riemann, tensors as a 6×6 matrix, in orthnormal coordinates, as was done in section 2, then an expression equivalent to Eqs. (3.3) but much simpler to use to calculate things, is given by

$$\widetilde{S} = H^T \, S \, H \ , \tag{3.9}$$

where S is the 6×6 matrix of components in an orthonormal basis, as given above in Eq. (2.1), while $\widetilde{S}$ is the 6×6 matrix of components in the associated null basis.

We may now present all those equalities between components of the conformal tensor, in this null basis, in the form of a 6×6 matrix for it, where we recall that since $\underset{\sim}{\theta}^2$ is the complex conjugate of $\underset{\sim}{\theta}^1$, i.e., $\underset{\sim}{\theta}^2 = \overline{\underset{\sim}{\theta}^1}$, we have relations such as $R_{1313} = \overline{R_{2323}}$.

$$C \ \Longrightarrow \ \begin{pmatrix} Q^{(3)} + \overline{Q^{(3)}} & \overline{Q^{(2a)}} & Q^{(2a)} & Q^{(3)} - \overline{Q^{(3)}} & -\overline{Q^{(4a)}} & Q^{(4a)} \\ \overline{Q^{(2a)}} & \overline{Q^{(1)}} & 0 & \overline{-Q^{(2b)}} & \overline{Q^{(3)}} & 0 \\ Q^{(2a)} & 0 & Q^{(1)} & Q^{(2b)} & 0 & -Q^{(3)} \\ Q^{(3)} - \overline{Q^{(3)}} & -\overline{Q^{(2b)}} & Q^{(2b)} & \overline{Q^{(3)}} + Q^{(3)} & -\overline{Q^{(4b)}} & Q^{(4b)} \\ -\overline{Q^{(4a)}} & \overline{Q^{(3)}} & 0 & \overline{-Q^{(4b)}} & \overline{Q^{(5)}} & 0 \\ Q^{(4a)} & 0 & -Q^{(3)} & Q^{(4b)} & 0 & Q^{(5)} \end{pmatrix} , \tag{3.10}$$

where the notation has been created so that the standard 5 Petrov scalars are easily determined from this:

$$\begin{gathered} C^{(5)} \equiv 2C_{2424} = 2Q^{(5)} \ , \quad C^{(3)} \equiv 2C_{4231} = 2Q^{(3)} \ , \quad C^{(1)} \equiv 2C_{3131} = 2Q^{(1)} \ , \\ C^{(4)} \equiv C_{1242} + C_{3442} = -Q^{(4a)} - Q^{(4b)} \ , \quad C^{(2)} \equiv C_{1231} + C_{3431} = Q^{(2a)} + Q^{(2b)} \ . \end{gathered} \tag{3.11}$$