Summary of Formulae for the Robertson-Walker Metric

It is helpful to begin with a few careful definitions of some mathematical notions associated with these metrics.

1. A (local) **diffeomorphism** is a mapping of some (open) neighborhood of a manifold M into itself that is invertible and both it and its inverse are differentiable.

2. A (local) **isometry** is a (local) diffeomorphism that also preserves the metric structure of the manifold within that neighborhood.

3. A space is **homogeneous** in some neighborhood, U, if any point, $P \in U \subseteq M$ can be mapped by an isometry into any other point, $Q \in U \subseteq M$.

4. A space is **isotropic** in some neighborhood, $U \subseteq M$, if at every point $P \in U$, there exists an isometry that leaves the point P fixed, but which maps any basis vector in $\leq T_P$ into any other such basis vector.

Note: Isotropy implies the existence of the maximal number of independent isometries possible.

Note: **Isotropy about every point implies homogeneity**

5. A **spatially homogeneous spacetime** is one which is *foliated by* a one-parameter family, Σ_t, of spacelike hypersurfaces, each one of which is homogeneous. By *foliated*, we mean that the entire spacetime is made up of "leaves,"—the many Σ_t—all "stacked up" on top of each other, as the parameter t takes on all its allowed values.

6. A **spatially isotropic spacetime** is one which contains a congruence of timelike curves, with tangent $\widetilde{u}$, i.e., (co-moving) observers, that fill the spacetime and are such that for each point $P \in M$ the spacelike-directed curves orthogonal to $\widetilde{u}_P$, that correspond locally to directions to other such observers, fill a (locally) spacelike surface that is an isotropic 3-surface.

Presuming that the space is spatially isotropic everywhere, these spacelike surfaces are in fact just the spacelike hypersurfaces Σ_t discussed in the previous section. By adjusting their origins and scales, if necessary, this congruence of observers possesses a uniform proper time, which we refer to as "cosmic time," and use the variable t to refer to it.

7. The 3 metric(s) discovered, separately, by Friedmann, with Lemaitre involved, and again by Robertson and Walker describe all possible 4-dimensional spacetimes which are spatially isotropic (and homogeneous). They therefore require the existence, as above, of a foliation by spacelike hypersurfaces $\{\varSigma_t \mid t = -\infty \ldots +\infty\}$. On these surfaces, we denote the 3-dimensional, spatial metric there by $\boldsymbol{\sigma} \equiv \sigma_{ij}\, dx^i dx^j$, and note that there are only 3 possible such metrics, i.e., which are everywhere homogeneous and isotropic, all of which of course have constant curvature:

$$\begin{aligned} \text{positive curvature, a 3-sphere:} &\quad \boldsymbol{\sigma} = d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\ , \\ \text{zero curvature, flat Euclidean space:} &\quad \boldsymbol{\sigma} = d\psi^2 + \psi^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\ , \\ \text{negative curvature, a 3-hyperboloid:} &\quad \boldsymbol{\sigma} = d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\ . \end{aligned}$$

It is of course more usual to use the symbol r instead of ψ when the space is flat; nonetheless, this generates a uniformity in the appearance which has some value. Actually, let's do that, but let's do it somewhat more generically: We define a symbol r for each of these three allowed spaces:

$$d\psi \equiv \begin{cases} \frac{dr}{\sqrt{1-r^2}}\ , & \text{for the 3-sphere,} \\ dr\ , & \text{for flat 3-space,} \\ \frac{dr}{\sqrt{1+r^2}}\ , & \text{for the 3-hyperboloid,} \end{cases} \quad \Longrightarrow \quad r \equiv f(\psi) = \begin{cases} \sin\psi\ , & \text{for the 3-sphere,} \\ \psi\ , & \text{for flat 3-space,} \\ \sinh\psi\ , & \text{for the 3-hyperboloid.} \end{cases}$$

We may then insert the one or the other of these metrics into the notion above, of foliating the spacetime by homogeneous, isotropic 3-surfaces, and having normal, timelike geodesics for them. We can use a time-dependent scaling, with dimensions of length, that would multiply each 3-metric, which allows us to write the most general such metric in the following form:

$$\mathbf{g} \ = \ -dt^2 + R^2(t)\,\boldsymbol{\sigma} = \ -dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\, d\Omega^2\right\}\ , \qquad k = +1, 0, -1\,. \tag{1}$$

Sometimes it is also useful to introduce a different time variable, often referred to as "arc-time," which demonstrates explicitly that our metric is conformally equivalent to a simpler one. We define

$$d\eta \equiv dt/R(t)\ , \tag{1a}$$

and write the metric in the following alternative form, showing that it is conformally equivalent to flat space:

$$\mathbf{g} \;=\; R^2(t)\left\{-d\eta^2 + d\psi^2 + f^2(\psi)\, d\Omega^2\right\} \, . \tag{1b}$$

We define an orthonormal tetrad for the Robertson-Walker metric in the obvious way,

$$\underset{\sim}{\omega}^r \equiv R\, d\psi = R\frac{dr}{\sqrt{1-kr^2}} \;, \quad \underset{\sim}{\omega}^\theta \equiv R\, r\, d\theta \;, \quad \underset{\sim}{\omega}^\varphi \equiv R\, r \sin\theta\, d\varphi \;, \quad \underset{\sim}{\omega}^t = dt = R\, d\eta \, . \tag{2}$$

Then we may immediately calculate the connection 1-forms,

$$\begin{gathered} \underset{\sim}{\Gamma}_{r\theta} = -\frac{\sqrt{1-kr^2}}{r\,R}\,\underset{\sim}{\omega}^\theta \;, \quad \underset{\sim}{\Gamma}_{r\varphi} = -\frac{\sqrt{1-kr^2}}{r\,R}\,\underset{\sim}{\omega}^\varphi \;, \quad \underset{\sim}{\Gamma}_{\theta\varphi} = -\frac{\cot\theta}{r\,R}\,\underset{\sim}{\omega}^\varphi \;, \\ \underset{\sim}{\Gamma}_{rt} = \frac{\dot R}{R}\,\underset{\sim}{\omega}^r \;, \quad \underset{\sim}{\Gamma}_{\theta t} = \frac{\dot R}{R}\,\underset{\sim}{\omega}^\theta \;, \quad \underset{\sim}{\Gamma}_{\varphi t} = \frac{\dot R}{R}\,\underset{\sim}{\omega}^\varphi \;, \end{gathered} \tag{3}$$

and the curvature 2-forms,

$$\begin{gathered} \underset{\sim}{\Omega}_{r\theta} = \frac{\dot R^2 + k}{R^2}\,\underset{\sim}{\omega}^r \wedge \underset{\sim}{\omega}^\theta \;, \quad \underset{\sim}{\Omega}_{r\varphi} = \frac{\dot R^2 + k}{R^2}\,\underset{\sim}{\omega}^r \wedge \underset{\sim}{\omega}^\varphi \;, \quad \underset{\sim}{\Omega}_{\theta\varphi} = \frac{\dot R^2 + k}{R^2}\,\underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^\varphi \;, \\ \underset{\sim}{\Omega}_{rt} = -\frac{\ddot R}{R}\,\underset{\sim}{\omega}^r \wedge \underset{\sim}{\omega}^t \;, \quad \underset{\sim}{\Omega}_{rt} = -\frac{\ddot R}{R}\,\underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^t \;, \quad \underset{\sim}{\Omega}_{rt} = -\frac{\ddot R}{R}\,\underset{\sim}{\omega}^\varphi \wedge \underset{\sim}{\omega}^t \;, \end{gathered} \tag{4}$$

or one can describe the curvatures by the very simple forms:

$$R_{r\theta r\theta} = R_{r\varphi r\varphi} = R_{\theta\varphi\theta\varphi} = \frac{\dot R^2 + k}{R^2} \;, \quad R_{rtrt} = R_{\theta t\theta t} = R_{\varphi t\varphi t} = -\frac{\ddot R}{R} \, . \tag{5}$$

From this one calculates immediately the conformal and Einstein parts of the curvature:

$$\begin{gathered} C_{\mu\nu\lambda\eta} = 0 \;, \\ \mathcal{G}_{rr} = \mathcal{G}_{\theta\theta} = \mathcal{G}_{\varphi\varphi} = -\left\{2\frac{\ddot R}{R} + \frac{\dot R^2 + k}{R^2}\right\} \;, \quad \mathcal{G}_{tt} = 3\frac{\dot R^2 + k}{R^2} \;, \quad \mathcal{G}_{\mu\nu} = 0 \,, \;\; \mu \neq \nu \end{gathered} \tag{6}$$

Given that we want to solve Einstein's equations, it is worth noting that the structure we have is that in this frame,

a.) Einstein's tensor is diagonal,

b.) all 3 of the diagonal, spatial components of Einstein's tensor are equal, and c.) the temporal component is different, so that there are only two independent degrees of freedom in the Einstein tensor.

These properties are also exactly the distinguishing characteristics of the stress-energy tensor for a perfect fluid, characterized by its pressure, P, and its rest-energy density, ρ; therefore, if we wanted to set Einstein's tensor equal to some stress-energy tensor for some sort of matter, it would actually have to correspond to that for a perfect fluid. The two Einstein equations are then simply

$$\begin{aligned} \mathcal{G}_{\mu\nu} - \Lambda\, g_{\mu\nu} = 8\pi\, T_{\mu\nu}\ ; \\ 8\pi\, P + \Lambda = -\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2}\right) \\ 8\pi\, \rho - \Lambda = 3\frac{\dot{R}^2 + k}{R^2}\ , \end{aligned} \tag{7}$$

so that we could "identify" those parts of the curvature that act like pressure, and like energy density.

These equations may be re-written in various useful ways. One approach is

$$\begin{aligned} \frac{\ddot{R}}{R} &= -\frac{4\pi}{3}(\rho + 3P) - \frac{\Lambda}{3}\ , \\ H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 &= \frac{8\pi}{3}\rho - \frac{k}{R^2} - \frac{\Lambda}{3}\ . \end{aligned} \tag{7a}$$

An approach that has **recently** become common is to claim that one should re-define Λ and k so that they look like densities. First considering Λ, we may define $\rho_\Lambda \equiv -\Lambda/8\pi = -P_\Lambda$. (As shown below, in Eq. (8), if $\rho + P = 0$, then ρ is a constant, consistent with the fact that Λ is a constant.) This construction not only takes the (Friedmann) equation for the Hubble parameter, H, and puts it in a form "easier to remember," but also does the same thing for the acceleration equation:

$$\begin{aligned} \frac{\ddot{R}}{R} &= -\frac{4\pi}{3}[(\rho_{\rm tr} + 3P_{\rm tr}) + (\rho_\Lambda + 3P_\Lambda)]\ , \\ \left(\frac{\dot{R}}{R}\right)^2 &= \frac{8\pi}{3}(\rho_{\rm tr} + \rho_\Lambda) - k/R^2\ , \end{aligned} \tag{7b}$$

where we have used the subscript tr to indicate all the true matter, as opposed to the ones coming from the cosmological constant.

We may also divide the equation for the Hubble constant by itself, to create an equation for a collection of dimensionless quantities all of which must add to the value +1:

$$1 = \frac{\rho_{\rm tr}}{\rho_c} - \frac{k}{\dot{R}^2} - \frac{\Lambda}{H^2} \equiv \Omega_{\rm tr} + \Omega_k + \Omega_\Lambda \ , \qquad \rho_c \equiv H^2/(8\pi/3) \ . \tag{7c}$$

Separately, we can apply the Bianchi identity to the problem. We know that it says that the divergence of each side is equal to zero. Straightforward calculation shows that the 4 components of ${T^{\mu\nu}}_{;\nu} = 0$ amount to the three reasonably trivial equations $\nabla P = 0$, which just say that the pressure does not change in space, and then the fourth one

$$\begin{gathered} 3(\rho + P)\frac{\dot{R}}{R} + \dot{\rho} = 0 \ , \\ \text{or} \\ \frac{d}{dt}(\rho R^3) + P\,\frac{d}{dt}R^3 = 0 \qquad \Leftrightarrow \qquad 0 = \frac{d}{dt}E + P\frac{d}{dt}V \ , \end{gathered} \tag{8}$$

where the third version of the equation above should remind us of the *adiabatic expansion* of our ideal gas.

Were we to know the equation of state for this matter, i.e., the relationship $P = P(\rho)$, then we could presumably solve this equation, remove P, say, from the equations above, and have a simpler problem to deal with. Therefore, it is customary to divide the total energy density, ρ into contributions from matter, ρ_m and from radiation, ρ_r, where we presume that we do in fact know the equations of state for those sorts of materials: **We assume** that we can treat the two separately, i.e., that they do not interact, at least not in the last more than 10 billion years, and, furthermore, that the equations of state allow us to say that $P_m = 0$ while $P_r = \rho_r/3$. This allows to make statements about the dependence on $R(t)$ of these terms:

$$\rho_m R^3 = \text{a constant} \ , \qquad \rho_r R^4 = \text{a constant}$$

$$\Longrightarrow \rho(t) = \rho_m(t) + \rho_r(t) = \rho_{m0}\left(\frac{R_0}{R(t)}\right)^3 + \rho_{r0}\left(\frac{R_0}{R(t)}\right)^4 , P(t) = \frac{1}{3}\rho_{r0}\left(\frac{R_0}{R(t)}\right)^4$$

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \frac{8\pi}{3}\left\{\rho_{m0}\left(\frac{R_0}{R}\right)^3 + \rho_{r0}\left(\frac{R_0}{R}\right)^4\right\} = -\tfrac{1}{3}\Lambda \tag{9}$$

This shows us the dependence of H^2 on terms that depend on $1/R^n$, where n has the values 0, 2, 3, and 4. One can then think of this as an equation of the standard form for the motion of a particle in a potential well, that one takes in ordinary classical mechanics, where the $H^2 = (\dot{R}/R)^2$ term is a kinetic energy, the others are potential energies and the cosmological constant acts like a (constant) energy term.

On the other hand, it is often better not to look at this equation in this form, but, rather to use the "arc time," η, as a variable instead. From Eq. (1a) we have the relationship

$$\frac{d\eta}{dt} = \frac{1}{R} \implies dR/d\eta = R\dot{R} = R^2\frac{\dot{R}}{R} , \tag{10a}$$

so that multiplication of the so-called Friedmann equation, Eq. (9) above, by a factor of R^4 allows it to be written in the following form, which no longer has difficulties at $R = 0$, and is therefore much more useful, for instance, for computerized calculations, $dt/d\eta = R$ being the other member of a pair of calculations to determine R and t as functions of η:

$$\left(\frac{dR}{d\eta}\right)^2 = +B^2 + A\,R - k\,R^2 - \tfrac{1}{3}\Lambda\,R^4 , \tag{10b}$$

where A and B^2 are positive constants,

$$A \equiv 2\frac{4\pi}{3}\rho_{m_0}R_0^3 , \quad B^2 \equiv 2\frac{4\pi}{3}\rho_{r_0}R_0^4 , \tag{10c}$$

and the quantities with subscript 0 are evaluated at some particular time, presumably now.

The "history," i.e., the time evolution, of this metric can then be obtained by solving together the pair of parametrized differential equations, (10a) and (10b) above, for $R = R(\eta)$ and $t = t(\eta)$. In the geneeral case, when the cosmological constant, Λ, is non-zero, the analytical solution of this equation involves elliptic functions and/or numerical calculations, although one can get acquire some understanding of it by simply using the potential-well method of thought already mentioned. In particular, the "potential" shown in Eq. (9), as a function of R, goes to negative infinity as $R \to 0$, to zero as R goes to infinity, and has a maximum for finite R only when $k=+1$. As well, the 3 curves for the 3 allowed values of k are moderately similar, with the one for $k=-1$ below the one for $k=0$, which is below the one for $k=+1$. Depending on the value of Λ, for smaller Λ one has oscillatory behavior for all values of k, while for values of Λ so that the "energy"-term lies above the maximum values of the "potential" the value of $R(t)$ will simply continue to increase forever, with no extremum reached.

On the other hand the cases with $\Lambda = 0$ may be integrated fairly easily, giving the following results:

$$
\begin{gathered}
k = \ +1 \ \Rightarrow \ \begin{cases} R = A(1-\cos\eta) + B\sin\eta \ , \\ \\ \ t = A(\eta - \sin\eta) + B(1-\cos\eta) \ , \\ \\ \qquad \Longrightarrow \quad R(\eta=\pi) = 2A, \ \text{max. R if } A >> B \, . \end{cases} \\
k = \ -1 \ \Rightarrow \ \begin{cases} R = A(\cosh\eta - 1) + B\sinh\eta \ , \\ \\ \ t = A(\sinh\eta - \eta) + B(\cosh\eta - 1) \ , \\ \\ \qquad \Longrightarrow \quad R(t) \approx t \text{ for very long times.} \end{cases} \\
k = 0 \ \Rightarrow \ \begin{cases} R = B\eta + \frac{1}{2}A\eta^2 \ , \\ \\ \ t = \frac{1}{2}B\eta^2 + \frac{1}{6}A\eta^3 \ . \end{cases}
\end{gathered}
\tag{11}
$$

In all 3 cases we have the approximate behavior at very early times, so long as $B \neq 0$:

$$
R(t) \approx \sqrt{2Bt} + O(t) \ . \tag{12a}
$$

If, for some reason, we would have $B=0$, i.e., no "radiation," then there are other possibilities:

$$
\begin{aligned}
B = 0 = \Lambda \, , \ A \neq 0 \ &\Longrightarrow R \propto t^{2/3} + O(t) \ , \\
k = B = 0 = A \, , \ \Lambda \neq 0 \ &\Longrightarrow R = R_0 \, e^{\pm\sqrt{-\Lambda/3}\, t} \ .
\end{aligned}
\tag{12b}
$$

Additional Comments on the timelike geodesics of our cosmic observers:

Our initial assumption concerning the congruence of observers, who measure "cosmic time" as their own proper time, and for whom our spacelike foliations are the homogeneous and isotropic 3 surfaces, allows us to describe this congruence so that their 4-velocities are simply $\widetilde{e}_t$. We can understand this congruence much better if we ask the usual questions about how these observers see the other members of their congruence, which requires that we caculate the (arbitrary 3-direction) covariant derivative of the congruence, i.e., determine the expansion, rotation, and shear. This requires the covariant derivative of $\tilde{u}$. We will calculate this as follows:

$$B_{\mu\lambda} \equiv g_{\mu\eta}\nabla_\lambda u^\mu = g_{\mu\eta}\nabla_\lambda \delta^\mu_t = \Gamma_{\mu\, t\, \lambda} \; .$$

Consulting our table of connections we easily find that this matrix is diagonal, with

$$B_{rr} = B_{\theta\theta} = B_{\varphi\varphi} = \frac{\dot{R}}{R} \equiv H(t) \; , \qquad B_{tt} = 0 \; .$$

Therefor both the shear and the rotation (or twist) are zero, while the expansion (or dilation) is given by

$$\Theta = 3\frac{\dot{R}}{R} = 3H \; ,$$

showing that the Hubble parameter is a scale length, with its time dependence indicating the rate at which the "distance" between different cosmic observers is changing.

It is also very good that the twist is zero, since the requirement for a congruence $\tilde{u}$ to have a globally-defined hypersurface to which it is the normal is that $\tilde{u}\wedge d\tilde{u} = 0$, which is just the twist, the rotation by a different name, and approach for calculation.

Comments on the Geodesic Equations over this Manifold:

The equations for a geodesic, with tangent vector $\widetilde{u} = u^{\hat{\mu}}\,\widetilde{e}_{\hat{\mu}}$, and affine parameter τ, are the following, where we first recall the components of $\widetilde{u}$ in our current basis:

$$\begin{gathered} \widetilde{u} \; \Longrightarrow \quad \left(\frac{R}{\sqrt{1-kr^2}}\, r' \; , \; r\, R\, \theta' \; , \; r\, R\, \sin\theta\, \varphi' \; , \; t'\right) \; , \\ \mu \equiv -1 \text{ or } 0 = (\widetilde{u})^2 = (u^{\hat{r}})^2 + (u^{\hat{\theta}})^2 + (u^{\hat{\varphi}})^2 - (u^{\hat{t}})^2 \; , \end{gathered} \tag{13}$$

where of course the option in the last equation, for the value of the constant μ, depends on whether the geodesic is timelike or null. We may then write the necessary equations, with

$d/d\tau$ denoted by a prime:

$$
\begin{aligned}
(u^{\hat{r}})' - \frac{\sqrt{1-kr^2}}{r\,R}[(u^{\hat{\theta}})^2 + (u^{\hat{\varphi}})^2] + \frac{\dot{R}}{R}\,u^{\hat{r}}\,u^{\hat{t}} &= 0\,,\\
(u^{\hat{\theta}})' + \frac{\sqrt{1-kr^2}}{r\,R}\,u^{\hat{r}}\,u^{\hat{\theta}} - \frac{\cot\theta}{r\,R}\,(u^{\hat{\varphi}})^2 + \frac{\dot{R}}{R}\,u^{\hat{\theta}}\,u^{\hat{t}} &= 0\,,\\
(u^{\hat{\varphi}})' + \frac{\sqrt{1-kr^2}}{r\,R}\,u^{\hat{r}}\,u^{\hat{\varphi}} + \frac{\cot\theta}{r\,R}\,u^{\hat{\theta}}\,u^{\hat{\varphi}} + \frac{\dot{R}}{R}\,u^{\hat{\varphi}}\,u^{\hat{t}} &= 0\,,\\
(u^{\hat{t}})' + \frac{\dot{R}}{R}\,[(u^{\hat{r}})^2 + (u^{\hat{\theta}})^2 + (u^{\hat{\varphi}})^2] &= 0\,.
\end{aligned}
\tag{14}
$$

As we know this manifold has a large amount of symmetry, there are several constants of the motion that allow partial integrations of the geodesic equations:

$$
\begin{aligned}
\left\{R(t) r \sin\theta\, u^{\hat{\varphi}}\right\} &= C,\ \text{a constant},\\
\left\{R(t) r\, u^{\hat{\theta}}\right\}^2 + \left(\frac{C}{\sin\theta}\right)^2 &= L^2,\ \text{a constant},\\
\left\{R(t) u^{\hat{r}}\right\}^2 + \left(\frac{L}{r}\right)^2 &= Q^2,\ \text{a constant},\\
(u^{\hat{t}})^2 = \left(\frac{dt}{d\tau}\right)^2 &= -\mu + \frac{Q^2}{R^2(t)}\,,
\end{aligned}
\tag{15}
$$

where the constant μ is either -1, for timelike curves, or 0 for null curves.