Lie Derivatives and (Conformal) Killing Vectors

0. Motivations for Lie derivatives

On some manifold, $\mathcal{M}$, or at least in some neighborhood, $U \subseteq \mathcal{M}$, we are concerned with a congruence of curves, all with tangent vectors given by $\widetilde{\xi}$, that are important for some particular problem, for instance the motions of a physical system over time, beginning at various different, nearby initial positions that lie in U. The Lie derivative is then a method of making a comparison of the value that some geometric object has at some point along the curve relative to the value that it would have had if it had been "dragged" along that curve to the same point, where we will carefully define below what we mean by the "dragging" of a geometric object. The Lie derivative, along $\widetilde{\xi}$ of some object T, will be defined (below) as the first-order change of this type, i.e., the first non-trivial term in the (infinite) Taylor series describing the changes that occur when moved along the curve with this tangent vector, and will be denoted by $\mathcal{L}_{\widetilde{\xi}}(T)$.

Especially important for relativity theory is the behavior of the metric when moved along curves on a manifold. We will define a **conformal Killing vector**, $\widetilde{\xi}$, as a vector field on a manifold such that when the metric is dragged along the curves generated by $\widetilde{\xi}$ its Lie derivative is proportional to itself:

$$\underset{\widetilde{\xi}}{\mathcal{L}} g_{\mu\nu} = 2\chi \, g_{\mu\nu} \ , \tag{0.1}$$

for some scalar field χ.

The physical understanding of this requirement is that when the metric is dragged along some congruence of curves it remains itself **modulo some scale factor**, χ, which may, perhaps, vary from place to place on the manifold.

In the case that χ is zero, we refer to this as a *true* Killing vector, and, clearly the metric is left completely invariant as it is dragged along the curves with that true Killing vector as their tangent vector. An obvious example is dragging a spherically symmetric metric along a path of constant latitude on a sphere.

On the other hand, when the field χ is constant, but not zero, the associated Killing vector is said to be *homothetic*, and the metric is being changed by a (constant) scale factor as it moves along. This too is quite interesting from first principles; a simple example is a dilatation, where objects are, for instance, enlarged but otherwise unchanged as you proceed in some direction—for instance, toward (spacelike) infinity.
Lastly, when χ is actually a function on the manifold, then this is an actual conformal Killing vector, rather than the more special cases just considered above. The physical meaning of this case is not quite as obvious, but will be shown below to be related to certain invariances not of the metric but, rather, of the curvature of the manifold. [Killing vectors are named for a Norwegian mathematician named W. Killing, who first described these notions in 1892.]

In order to create an appropriate definition of the Lie derivative, with respect to some vector field, we must first back up quite a bit, to create enough "new" differential geometry to do this properly. As well, after doing that we will want to proceed forward from that definition, to discover the integrability conditions that the existence of a Killing vector puts on the connection and curvature of a manifold.

1. Maps between manifolds, and their associated pullbacks and pushforwards

On some manifold, $\mathcal{M}$, we first choose a point $P \in U \subseteq \mathcal{M}$ and a map, ϕ, from U to some other manifold, $\mathcal{N}$, which could be just a different region of our original manifold:

$$\phi : U \subseteq \mathcal{M} \;\rightarrow\; W \subseteq \mathcal{N} \,, \qquad Q \equiv \phi(P) \in W \subseteq \mathcal{N} \tag{1.1}$$

Since there are various different tensor spaces attached to the manifold in this neighborhood, and in particular at the point P, it is reasonable to suppose that there should be associated with this map of the manifolds a method of correlating tensors over $P \in \mathcal{M}$ to tensors over $Q \equiv \phi(P) \in \mathcal{N}$. We intend to explain how this happens in some detail, but will find it convenient to begin with the simplest sort of tensors, i.e., functions which are tensors of $\left[{0 \atop 0}\right]$. Therefore, we now define the pullback of functions defined over $\mathcal{N}$ to those defined over $\mathcal{M}$.

Let f be a function defined over the neighborhood $W \subseteq \mathcal{N}$, i.e., $f : W \subseteq \mathcal{N} \to \mathbb{R}$. Then we may define an associated function $\phi^* f : U \subseteq \mathcal{M} \to \mathbb{R}$, the *pullback of f via ϕ* as the following:

$$\begin{array}{c}\textit{pullback}\\ \text{of functions}\end{array} : \begin{cases} \text{for } \phi : U \subseteq \mathcal{M} \to W \subseteq \mathcal{N} \text{ and } f : \mathcal{N} \to \mathbb{R}\,, \\ \phi^* f : U \subseteq \mathcal{M} \to \mathbb{R} \text{ so that } \forall S \in U \subseteq \mathcal{M},\ (\phi^* f)(S) \equiv f(\phi(S))\,. \end{cases} \tag{1.1}$$

Since individual coordinates are just functions, this allows me to use a coordinate system near $Q \in \mathcal{N}$ to define a coordinate system near $P \in \mathcal{M}$, i.e., to *pullback* the coordinate chart near $Q \in \mathcal{N}$ to create a (very strongly associated) coordinate chart near $P \in \mathcal{M}$.
Let $\{y^\alpha\}_1^n$ be a coordinate system defined over $W \subseteq \mathcal{N}$; then $\{x^\mu \equiv \phi^*(y^\alpha)\}_1^m$ is a proper coordinate system defined over $U \subseteq \mathcal{M}$.

Do note that it is not necessary that the dimensions of the two manifolds be the same, and we will shortly give examples where this is in fact true. **On the other hand,** all of our usages of these ideas to create the Lie derivative will in fact relate to the case when the two dimensions are in fact the same, and, in fact, to the case where the two manifolds are the same, even though the two neighborhoods will be different.

Now, since tangent vectors act as operators on functions, this allows us to define the *pushforward* of tangent vector fields:

$$\begin{array}{c}\textit{pushforward}\\ \text{of vectors}\end{array} : \begin{cases} \text{for } \phi : U \subseteq \mathcal{M} \to W \subseteq \mathcal{N} \text{ and } \widetilde{v} \in \boldsymbol{\mathfrak{T}}_{|U}\,,\ \widetilde{v} : \boldsymbol{\mathfrak{F}}_{|U} \to \boldsymbol{\mathfrak{F}}_{|U} \\ (\phi_* \widetilde{v}) \in \boldsymbol{\mathfrak{T}}_{|W} \text{ so that } \forall f \in \boldsymbol{\mathfrak{F}}_{|W}\,,\quad (\phi_* \widetilde{v})(f) \equiv \widetilde{v}(\phi^* f)\,. \end{cases} \tag{1.2}$$

To see that there actually is something to this definition, let us calculate the relationship between the components of the two vector fields, relative to their appropriate coordinate systems, $\{y^\mu\}$ and $\{x^\mu\}$, as defined above, noting that in coordinates, we may think of f as a function of the coordinates y^μ, while $\phi^* f$ is a function of the coordinates x^μ:

$$\begin{aligned} (\phi_* \widetilde{v})(f) = (\phi_* \widetilde{v})^\alpha \frac{\partial}{\partial y^\alpha}(f) = (\widetilde{v})^\nu \frac{\partial}{\partial x^\nu}(\phi^* f) = \widetilde{v}^\nu \frac{\partial y^\alpha}{\partial x^\nu} \frac{\partial}{\partial y^\alpha}(f) \\ \Longrightarrow (\phi_* \widetilde{v})^\alpha = v^\nu \frac{\partial y^\alpha}{\partial x^\nu}\,. \end{aligned} \tag{1.3}$$

It is worth remarking that this appears to be exactly the same as the transformation that happens to the components of a vector when one performs a coordinate transformation at a single point on the manifold. However, it is really quite different philosophically because we are comparing components of vectors acting on two different manifolds. There is nonetheless a reason for this apparent equivalence having to do with the difference between thinking of a coordinate transformation in a *passive mode* or in an *active mode*.

a). The passive mode thinks of transformations of the coordinate charts at each point of some neighborhood, leaving of course the actual points themselves unchanged.

b). The active mode thinks of transformations of the points themselves of a manifold, and our method created above for mapping charts into other charts sort of drags the coordinate charts along.

Continuing, now having a method to move tangent vectors between the two manifolds, and also functions, we may easily perform the reverse sort of mapping for 1-forms, remembering that 1-forms map tangent vectors to real numbers: the *pullback* of a 1-form, $\underset{\sim}{\alpha} \in \Lambda^1_{|W}$ is denoted by $\phi^* \underset{\sim}{\alpha} \in \Lambda^1_{|U}$:

$$\begin{array}{c} \textit{pullback} \\ \text{of 1-forms} \end{array} : \begin{cases} \text{for } \phi : U \subseteq \mathcal{M} \to W \subseteq \mathcal{N} \text{ and } \underset{\sim}{\alpha} \in \Lambda^1_{|W} \;,\; \underset{\sim}{\alpha} : \boldsymbol{\mathcal{T}}_{|W} \to \mathbb{R} \\ \phi^* \underset{\sim}{\alpha} : \Lambda^1_{|U} \to \mathbb{R} \text{ so that } \forall\, \widetilde{v} \in \mathcal{T}_{|U} \;, \quad (\phi^* \underset{\sim}{\alpha})(\widetilde{v}) \equiv \underset{\sim}{\alpha}(\phi_* \widetilde{v}) \;. \end{cases} \tag{1.4}$$

We now want the relationship between the two sets of components, which we acquire by following along the same line of reasoning as was used to determine the relationship between the two sets of components of the vector fields:

$$\begin{aligned} (\phi^* \underset{\sim}{\alpha})_\mu \, v^\mu = (\phi^* \underset{\sim}{\alpha})(\widetilde{v}) \; &\equiv \underset{\sim}{\alpha}\,(\phi_* \widetilde{v}) = \alpha_\beta (\phi_* \widetilde{v})^\beta = \alpha_\beta v^\mu \frac{\partial y^\beta}{\partial x^\mu} \\ \Longrightarrow \quad (\phi^* \underset{\sim}{\alpha})_\mu \; &= \alpha_\beta \frac{\partial y^\beta}{\partial x^\mu} \;. \end{aligned} \tag{1.5}$$

We see that it is exactly the same (Jacobian) matrix that is involved in the transformation as for tangent vectors. However, the difference in the two behaviors is the way the matrix is multiplied, on the left in the one case and on the right in the other, which is again analogous to the way in which contravariant and covariant objects in general are transformed.

Now we have sufficiently many definitions to explain why it is that one can pullback differential forms (including functions, which are after all just 0-forms) but must pushforward tangent vectors. If we remember that we have created the coordinates on $\mathcal{M}$, $\{x^\mu\}_1^m$, as dependent on the mapping ϕ and the coordinates $\{y^\alpha\}_1^n$ on $\mathcal{N}$, this suggests that, for any particular actual function performing the mapping we should be able to write out explicitly the $x^\mu = x^\mu(y^\alpha)$. Therefore, in particular then the chain rule of multi-variable calculus allows us to write the following relationships:

$$\frac{\partial}{\partial y^\alpha} = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial}{x^\mu} \,, \qquad dx^\mu = \frac{\partial x^\mu}{\partial y^\alpha} \, dy^\alpha \,, \tag{1.6}$$

where of course the chain rule treats these two sorts of objects quite differently. Comparing with the transformations of components for tangent vectors—which have basis vectors as partial derivatives with respect to the coordinates—and the components for 1-forms—which have basis vectors as differentials of the coordinates—we see that they transform in exactly the opposite ways, in both cases, so that those transformation relationships are actually just rather advanced formulations of the chain rule, on objects defined over manifolds.

2. Definitions for the Lie derivative of tensor fields

The Lie derivative is a method to determine how vector fields are changing in vector spaces over nearby points. It is different from the covariant derivative, which relies on the specification of an affine connection, the function of which is to describe, locally, how the choice of a basis set for vectors changes from point to point over a manifold. Instead, the Lie derivative relies on the behavior of the vector field which determines a congruence of (non-intersecting) curves defined in the neighborhood of a point.

To describe this derivative, we continue the discussion in §1, but now specialize to the case that the manifold $\mathcal{N}$ is indeed the same as $\mathcal{M}$, and that there is a given vector field, $\widetilde{\xi}$, defined over some neighborhood that includes both U and W, which might overlap, at least to some extent. Taking a parameter λ along each of a family of curves with tangent vector

$\widetilde{\xi}$, we may create curves which begin at some arbitrary point R near $P \in U$, which we may then denote by $\Gamma(\lambda; R) = e^{\lambda\widetilde{\xi}}R$. We can think of the set of these curves, through points in U, as constituting an example of the map ϕ between manifolds, as described above, mapping, for instance P into Q, which we will then note by the symbol ϕ_λ, since there is a different such map for each value of λ. In particular, ϕ_0 is then the identity map, and $\phi_{-\lambda} = (\phi_\lambda)^{-1}$, i.e., the inverse map to ϕ_λ, exists and it is just $e^{-\lambda\widetilde{\xi}}$. In general, this says that we may think of a set of curves defined over some neighborhood either as

i.) a congruence of maps from $\lambda \in \mathbb{R}$ to the manifold, parameterized by the (nearby) point on the manifold at which they begin, **in the passive mode as described above**, or

ii.) a family of maps from the manifold to itself, parameterized by the real number λ, which is then **an active mode view** of the same set of functions. In this case they are referred to as a family of *flows* (of the manifold), and we can think of them in the usual approach, via Taylor series, as defining the chart of coordinates to be used along the flow in terms of the one where the flow began:

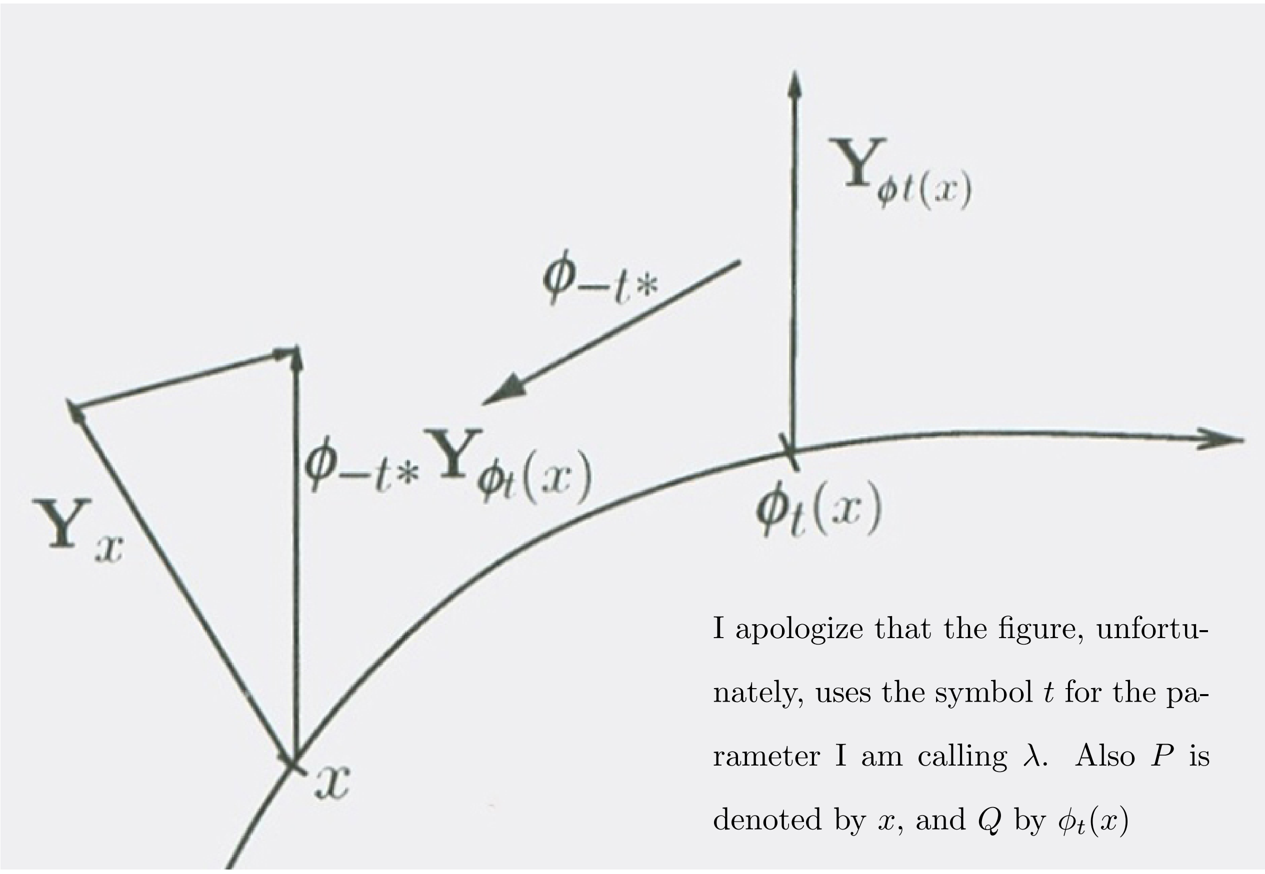
$$y^\mu = x^\alpha \delta^\mu_\alpha + \lambda \frac{d}{d\lambda}\phi^\mu_\lambda(P) + \tfrac{1}{2}\lambda^2 \frac{d^2}{d\lambda^2}\phi^\mu_\lambda(P) + \ \ldots\ ];, \tag{2.1}$$

where evaluating the derivatives at the point P means the same as saying that they are evaluated at $\lambda = 0$.

Since these maps are completely invertible, the use of these maps allows us to pullback or pushforward whatever tensor we want between, for instance, the tensor spaces over P and those over Q, because we can always use the inverse of the maps if that is what is needed. For example, let us consider two tensors of $\left[{1 \atop 1}\right]$, $\mathbf{T}$ defined at P and $\mathbf{B}$ defined at Q. Then we may define the pushforward value for $\mathbf{T}$, namely $\phi_{*\lambda}\mathbf{T}$, as a tensor over Q, and, as well, the pullback value for $\mathbf{B}$, namely $\phi^*_\lambda\mathbf{B}$, a tensor over P by the following schemes where we note that one defines a tensor by giving its action on the appropriate numbers of tangent vectors and 1-forms. Since these are $\left[{1 \atop 1}\right]$ tensors, we need to have available over the manifold at the point Q a 1-form field and a tangent vector field, $\underset{\sim}{\alpha} \in \Lambda^1_{|Q}$ and $\widetilde{v} \in \mathcal{T}_{|Q}$, and then the same sort of objects over the point P, namely $\underset{\sim}{\beta} \in \Lambda^1_{|P}$ and $\widetilde{w} \in \mathcal{T}_{|P}$. This allows us then to define the *pushforward of* $\mathbf{T}$ and the *pullback of* $\mathbf{B}$ as follows:

$$
\begin{aligned}
\underset{\text{at Q}}{(\phi_{*\lambda}\mathbf{T})}(\underset{\sim}{\alpha}, \widetilde{v}) \equiv \underset{\text{acting at P}}{\mathbf{T}(\phi^*_\lambda \underset{\sim}{\alpha}, (\phi^{-1}_\lambda)_* \widetilde{v})} \quad &\Longrightarrow \quad (\phi_{*\lambda}\mathbf{T})^\mu{}_\alpha = \left(\frac{\partial y^\mu}{\partial x^\nu}\right)\left(\frac{\partial x^\beta}{\partial y^\alpha}\right)\mathbf{T}^\nu{}_\beta \ ; \\
\underset{\text{at P}}{(\phi^*_\lambda\mathbf{B})}(\underset{\sim}{\sigma}, \widetilde{w}) \equiv \underset{\text{acting at Q}}{\mathbf{B}((\phi^{-1}_\lambda)^* \underset{\sim}{\sigma}, \phi_{*\lambda}\widetilde{w})} \quad &\Longrightarrow \quad (\phi^*_\lambda\mathbf{B})^\nu{}_\beta = \left(\frac{\partial x^\nu}{\partial y^\mu}\right)\left(\frac{\partial y^\alpha}{\partial x^\beta}\right)\mathbf{B}^\mu{}_\alpha \ .
\end{aligned}
\tag{2.2}
$$

Now we consider the case where our tensor is defined over an entire neighborhood including both P and Q, and the curve with tangent vector $\widetilde{\xi}$ joining them, and $\mathbf{Y}$ is a tensor defined over this entire neighborhood. We may now take this tensor as it is defined at Q and pull it back to the tensor space over the point P. Will this pullback be equal to the original tensor field $\mathbf{Y}$ defined in the tensor space over the point P. The answer is "probably not!" See the figure below to visualize this better:



I apologize that the figure, unfortunately, uses the symbol t for the parameter I am calling λ. Also P is denoted by x, and Q by $\phi_t(x)$

Both the tensors $\phi^*_{-\lambda}(\mathbf{Y}_{|_{\phi_\lambda(P)}})$ and $\mathbf{Y}_{|_P}$ lie in the same tensor space, defined over P; therefore, we may certainly ask for their difference, which depends on λ, and is shown in the figure as the vector which is the difference of these two quantities, both in the tensor space over P, which we denote as

$$\Delta_\lambda \mathbf{Y}_{|_P} \equiv \phi^*_{-\lambda}(\mathbf{Y}_{|_{\phi_\lambda(P)}}) - \mathbf{Y}_{|_P} \ .$$

We may then ask, for the given curve $\widetilde{\xi}$, what is the first-order change of this sort, which we will call the Lie derivative in the direction $\widetilde{\xi}$. More precisely, we define the following tensor in the (appropriate) tensor space at P:

$$\mathop{\mathcal{L}}_{\widetilde{\xi}} \mathbf{Y}_{|_P} \equiv \lim_{\lambda\to 0} \frac{1}{\lambda} \left\{ [\phi^*_{-\lambda}(\mathbf{Y}_{|_{\phi_\lambda(P)}})]_{|_P} - \mathbf{Y}_{|_P} \right\} \ . \tag{2.3}$$

We shall see below that this is indeed a tensor of the same type as $\mathbf{Y}$.

To evaluate this geometrically-based idea in terms of components, we first take the coordinates of P as x^μ, and then recall that $Q = \phi_\lambda(P)$ lies along the path with tangent vector $\widetilde{\xi}$, so that

$$\begin{aligned} y^\mu \equiv \phi^\mu_\lambda[x(P)] &= x^\mu(P) + \lambda \frac{d}{d\lambda}\phi^\mu_\lambda(P) + \tfrac{1}{2}\lambda^2 \frac{d^2}{d\lambda^2}\phi^\mu_\lambda(P) + \ \dots \\ &= x^\mu(P) + \lambda \xi^\mu(\lambda)\big|_P + \tfrac{1}{2}\lambda^2 \frac{d}{d\lambda}\xi^\mu(\lambda)|_P + \dots \ , \end{aligned} \tag{2.1$'$}$$

This allows us to think of the tensor fields in terms of a functional dependence on the coordinates of the points at which they exist, so that we can represent $\mathbf{Y}_P$ as $\mathbf{Y}[x(P)]$. Since we are comparing two different vectors in the same vector space, with respect to the same basis set, it is simplest if we look at the problem in terms of its components. Therefore, let us suppose, again, the special case that $\mathbf{Y}$ is a $\left[{1 \atop 1}\right]$ tensor, so that its components are $Y^\nu{}_\beta$, which gives us

$$Y^\nu{}_{\beta_{|_Q}} \implies Y^\nu{}_\beta(x + \lambda\xi) = Y^\nu{}_\beta(x) + \lambda\, \widetilde{\xi}(Y^\nu{}_\beta)(x) + O(\lambda^2) \ . \tag{2.4a}$$

As well we have to effect the tensor transformations described above in Eq. (2.2). Since $\{y^\mu\}$ are coordinates at Q, along the curve with tangent vector $\widetilde{\xi}$, we have

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu + \lambda\, \xi^\mu_{,\nu} + O(\lambda^2) \ , \qquad \frac{\partial x^\nu}{\partial y^\mu} = \delta^\nu_\mu - \lambda\, \xi^\nu_{,\mu} + O(\lambda^2) \ . \tag{2.4b}$$

Applying the second line of Eqs. (2.2), for $\phi^*_{-\lambda}$, to the first term in Eq. (2.4a), and bringing along the second term there, and inserting this into the definition given in Eq. (2.3), taking the limit gives us

$$\mathcal{L}_{\widetilde{\xi}} Y^\mu{}_\nu = \xi^\eta \partial_\eta Y^\mu{}_\nu - \xi^\mu_{,\eta} Y^\eta{}_\nu + \xi^\eta_{,\nu} Y^\mu{}_\eta \, . \tag{2.4c}$$

More or less, simply as a mnemonic for remembering how this formulation works, we compare its form to that of the covariant derivative of a tensor, $\mathbf{Y}$, along the same direction, $\widetilde{\xi}$, which is

$$\nabla_{\widetilde{\xi}} Y^\mu{}_\nu = Y^\mu{}_{\nu,\eta}\xi^\eta + (\Gamma^\mu{}_{\rho\eta}\xi^\eta) Y^\rho{}_\nu - (\Gamma^\rho{}_{\nu\eta}\xi^\eta) Y^\mu{}_\rho \, .$$

We see that the purely-algebraic format is to replace the contraction of the direction with the connection, i.e., $\underset{\sim}{\Gamma}^\mu{}_\rho(\widetilde{\xi}) = \Gamma^\mu{}_{\rho\eta}\xi^\eta$, by the negative of the derivative of the direction in which the Lie derivative is being taken, i.e., by $-\xi^\mu{}_{,\rho}$.

It is then useful to go ahead and write down some specific, important examples, and also to present them in both a form that is independent of choice of basis and one involving coordinates explicitly. Therefore we begin with just the action on a function:

$$\text{for} \quad f \in \mathcal{F}, \quad \mathcal{L}_{\widetilde{\xi}} f = \widetilde{\xi}(f) = \xi^\eta \partial_\eta f = \xi^\alpha f_{,\alpha} = \nabla_{\widetilde{\xi}} f \, , \tag{2.5}$$

so that we see—as is hopefully logical—that there are no differences between the Lie derivative, the ordinary derivative operator, and the covariant derivative when acting on just a function.

We can, however, now go further and look again at the action on a vector field, say $\widetilde{v}$, and also on a 1-form, say $\underset{\sim}{\alpha}$:

$$\begin{aligned} &\text{for} \quad \widetilde{v} \in \mathcal{T}, \quad \mathcal{L}_{\widetilde{\xi}} \widetilde{v} = \widetilde{e}_\mu \left(\xi^\eta v^\mu{}_{,\eta} - \xi^\mu{}_{,\eta} v^\eta \right) = [\widetilde{\xi} \, , \widetilde{v}] \, ; \\ &\text{for} \quad \underset{\sim}{\alpha} \in \Lambda^1 \, , \quad \mathcal{L}_{\widetilde{\xi}} \underset{\sim}{\alpha} = \underset{\sim}{\omega}^\nu \left\{ \xi^\mu \alpha_{\nu,\mu} + \xi^\mu{}_{,\nu} \alpha_\mu \right\} \, . \end{aligned} \tag{2.6}$$

It is quite convenient that the form for the Lie derivative of a tangent vector simply takes the form of the commutator of the two vector fields, which is completely coordinate- and basis-independent. It would be very good if we also had a form like that for the Lie derivative of a

1-form, which we now want to obtain. However, to obtain that form it is convenient to first define the action of an arbitrary p-form on a single vector. We know that a p-forms needs p vectors in order to give a number; therefore, if it is given just 1 vector it remains a $p-1$-form, i.e., it wants that many more vectors yet. In principle, one also needs to tell such a product just where, in the list of p different desired vectors, this given vector will be placed. There are two reasonably common places where it might be put, namely all the way to the left, or the right, in the list—the only ones available for a 2-form. Therefore the so-called *interior product*, or *step product* of a vector and a p-form has been created for this purpose:

$$\widetilde{v} \lrcorner \, \underset{\sim}{\beta} \equiv \underset{\sim}{\beta}(\widetilde{v}, \cdot, \ldots, \cdot) \; , \qquad \underset{\sim}{\beta} \llcorner \, \widetilde{v} \equiv \underset{\sim}{\beta}(\cdot, \ldots, \cdot, \widetilde{v}) \; , \tag{2.7}$$

where in each case the entries of $\cdot$ in the list of allowed objects for the p-form to act on has simply not yet been given.

With that information in hand, now let's consider the following calculations, using a coordinate basis set:

$$\begin{aligned} d[\underset{\sim}{\alpha}(\widetilde{v})] = d(\alpha_\mu v^\mu) = (\alpha_{\mu,\rho} v^\mu + \alpha_\mu v^\mu{}_{,\rho}) dx^\rho \; , \\ \widetilde{v} \lrcorner \, d\underset{\sim}{\alpha} = d\underset{\sim}{\alpha}(\widetilde{v}, \cdot) = v^\mu (\alpha_{\rho,\mu} - \alpha_{\mu,\rho}) dx^\rho \; , \\ \Longrightarrow \quad d[\underset{\sim}{\alpha}(\widetilde{v})] + \underset{\sim}{\alpha}(\widetilde{v}, \cdot) = (v^\mu \alpha_{\rho,\mu} + \alpha_\mu v^\mu{}_{,\rho}) dx^\rho = \underset{\widetilde{v}}{\mathcal{L}} \underset{\sim}{\alpha} \; , \end{aligned} \tag{2.8}$$

so that the last line of the equation just above provides us with a coordinate- and basis-free way of calculating the Lie derivative of a 1-form, if and when that may be needed.

In particular this form allows us to prove a very useful theorem, namely that

The Lie derivative and the exterior derivative commute when acting on p-forms.

$$\begin{aligned} \underset{\widetilde{\xi}}{\mathcal{L}} \underset{\sim}{\beta} = \widetilde{\xi} \lrcorner \, d\underset{\sim}{\beta} + d(\widetilde{\xi} \lrcorner \, \underset{\sim}{\beta}) \; , \\ \Longrightarrow \quad \underset{\widetilde{\xi}}{\mathcal{L}} d\underset{\sim}{\beta} = d(\widetilde{\xi} \lrcorner \, d\underset{\sim}{\beta}) = d \left\{ \underset{\widetilde{\xi}}{\mathcal{L}} \underset{\sim}{\beta} - d(\widetilde{\xi} \lrcorner \, \underset{\sim}{\beta}) \right\} = d \, \underset{\widetilde{\xi}}{\mathcal{L}} \underset{\sim}{\beta} \; . \end{aligned} \tag{2.9}$$

It is now also a desirable additional theorem to see that the definition of the Lie derivative is unchanged when we exchange the partial derivatives in its definition for covariant derivatives.

The proof of this is somewhat complicated; therefore, to simplify the atual algebra, I will perform the associated calculations in an ordinary (holonomic) coordinate basis for tangent vectors and 1-forms. Nonetheless, since the final results will use covariant derivatives in a totally basis-independent format, the final results are valid in any choice of basis whatsoever. We give this particular example for our standard $\left[{1 \atop 1}\right]$ tensor, $\mathbf{Y}$, so that we see the method for both contravariant and covariant indices:

$$\begin{aligned}\xi^\eta Y^\mu{}_{\nu;\eta} - \xi^\mu{}_{;\eta} Y^\eta{}_\nu + \xi^\eta{}_{;\nu} Y^\mu{}_\eta &= \xi^\eta Y^\mu{}_{\nu,\eta} - \xi^\mu{}_{,\eta} Y^\eta{}_\nu + \xi^\eta{}_{,\nu} Y^\mu{}_\eta \\ &\quad + \xi^\eta \left(\Gamma^\mu{}_{\lambda\eta} Y^\lambda{}_\nu - \Gamma^\lambda{}_{\nu\eta} Y^\mu{}_\lambda\right) - \Gamma^\mu{}_{\lambda\eta}\xi^\lambda Y^\eta{}_\nu + \Gamma^\lambda{}_{\eta\nu}\xi^\eta Y^\mu{}_\lambda \\ &= \xi^\eta \partial_\eta Y^\mu{}_\nu - \xi^\mu{}_{,\eta} Y^\eta{}_\nu + \xi^\eta{}_{,\nu} Y^\mu{}_\eta = \underset{\widetilde{\xi}}{\mathcal{L}} Y^\mu{}_\nu \ ,\end{aligned} \tag{2.10}$$

where we can see that all the terms with components of the connection simply cancel out.

As it is a very important special case of a tensor, which defines a Killing vector as was noted on the first page of these notes, let us write out explicitly the Lie derivative of the metric, using covariant derivatives:

$$\underset{\widetilde{\xi}}{\mathcal{L}} g_{\mu\nu} = \xi^\eta g_{\mu\nu;\eta} + \xi^\eta{}_{;\mu} g_{\eta\nu} + \xi^\eta{}_{;\nu} g_{\mu\eta} = (\xi^\eta g_{\eta\nu})_{;\mu} + (\xi^\eta g_{\mu\eta})_{;\nu} = \xi_{(\mu;\nu)} \ , \tag{2.11}$$

which shows the fairly standard method of performing actual calculations looking for Killing vectors, where the Lie derivative of the metric is simply proportional, again, to the metric.

Next we should consider the Lie derivative of the components of the connection. Since they are not tensor indices, we should expect them to have a different form for the Lie derivative:

$$\begin{aligned}\phi^*_\lambda \Gamma^\mu{}_{\nu\lambda|_P} &= X^\mu{}_\tau (X^{-1})^\eta{}_\nu (X^{-1})^\sigma{}_\lambda \Gamma^\tau{}_{\eta\sigma|_P} + X^\mu{}_\tau (X^{-1})^\tau{}_{\nu,\lambda|_P} \ , \\ \Longrightarrow \ \underset{\widetilde{\xi}}{\mathcal{L}} \Gamma^\mu{}_{\nu\lambda} &= \xi^\eta\, \Gamma^\mu{}_{\nu\lambda,\eta} - \xi^\mu_{,\tau}\Gamma^\tau{}_{\nu\lambda} + \xi^\tau_{,\nu}\Gamma^\mu{}_{\tau\lambda} + \xi^\tau_{,\lambda}\Gamma^\mu{}_{\nu\tau} + (\xi^\mu_{,\nu})_{,\lambda} \ .\end{aligned} \tag{2.12a}$$

Now we rewrite this second derivative in terms of a second covariant derivative plus whatever extra terms are needed, and insert it into the calculation above:

$$\xi^\mu{}_{;\nu\lambda} = \xi^\mu{}_{;\nu,\lambda} + \Gamma^\mu{}_{\sigma\lambda}\xi^\sigma{}_{;\nu} - \Gamma^\sigma{}_{\nu\lambda}\xi^\mu{}_{;\sigma} = \xi^\mu_{,\nu\lambda} + \Gamma^\mu{}_{\sigma\nu,\lambda}\xi^\sigma + \Gamma^\mu{}_{\sigma\nu}\xi^\sigma_{,\lambda} + \Gamma^\mu{}_{\sigma\lambda}\xi^\sigma{}_{;\nu} - \Gamma^\sigma{}_{\nu\lambda}\xi^\mu{}_{;\sigma} \tag{2.12b}$$

This gives us two pairs of terms which are simply the difference of the covariant derivative and the ordinary derivative, which we replace by the form involving the connection, which results in the following quite interesting result, involving the curvature tensor:

$$\underset{\widetilde{\xi}}{\mathcal{L}}\Gamma^{\mu}{}_{\nu\lambda} = \xi^{\mu}{}_{;\nu\lambda} + \xi^{\eta}\left\{\Gamma^{\mu}{}_{\nu[\lambda,\eta]} + \Gamma^{\mu}{}_{\tau[\eta}\Gamma^{\tau}{}_{\nu\lambda]}\right\} = \xi^{\mu}{}_{;\nu\lambda} - \xi^{\eta}R^{\mu}{}_{\nu\lambda\eta}\ . \tag{2.12c}$$

Since the curvature tensor has arisen, it is reasonable that we should also ask about its Lie derivative. Following the rules above for arbitrary tensors, we may immediately write that down:

$$\underset{\widetilde{\xi}}{\mathcal{L}}R_{\mu\nu\lambda\eta} = \xi^{\tau}R_{\mu\nu\lambda\eta;\tau} + 2\xi^{\tau}{}_{;[\mu}R_{\tau\nu]\lambda\eta} + \xi^{\tau}{}_{;[\lambda}R_{\mu\nu\tau\eta]}\ , \tag{2.13}$$

so that the equations above give us the form for the Lie derivative of all the quantities of interest for Killing vectors.

3. Implications from the existence of a Killing Vector

The existence of a Killing vector tells us immediately about the symmetries of the metric. If, for instance, $\widetilde{K} = \partial/\partial q$ is a (true) Killing vector for some manifold, then it is always possible to arrange a coordinate system, with q as one of the coordinates, so that the components of the metric with respect to that coordinate basis do not depend on q. It should be noted that this can be done for any collection of Killing vectors **provided** that they commute with one another; otherwise, one must choose whichever one is desired. Therefore, for instance, in the static, spherically-symmetric metric, we may choose coordinates so that $g_{\mu\nu}$ are independent of φ and t, but may not also eliminate the dependence on θ.

To think a little bit more about that necessary dependence on θ, we note that if the commutator of a pair of Killing vectors is not zero then the tangent vector that is defined by that commutator is also a Killing vector. This is, in principle, a very useful way to generate not-yet-found Killing vectors for a given metric.

Another very useful feature of Killing vectors is that they may be used to determine "constants of the motion," i.e., quantities that will be constant along any given geodesic. To see this, consider a Killing vector $\widetilde{K}$ and a 4-velocity $\widetilde{u}$, tangent to some geodesic motion. Then it is claimed that the scalar quantity $\widetilde{K}\cdot\widetilde{u}\equiv \mathbf{g}(\widetilde{K},\widetilde{u})$ is constant along that motion:

$$\widetilde{u}(\widetilde{K}\cdot\widetilde{u}) = u^\mu\nabla_\mu(K^\eta g_{\eta\lambda}u^\lambda) = K^\eta\, u^\mu\nabla_\mu u_\eta + u^\lambda u^\mu\nabla_\mu K_\lambda = u^\lambda u^\mu K_{(\lambda;\mu)} = 0\ . \tag{3.1}$$

4. Further conditions for the existence of a (conformal) Killing Vector

The symmetries that allow the existence of a Killing vector also put constraints on the connections and curvatures of the manifold; or the values of the connections and curvatures put additional constraints on the existence of a Killing vector, as one chooses to proceed. We now proceed, therefore, to resolve the complete set of such constraints, which we think of as "integrability conditions" for the existence of a Killing vector. For some given metric, the determination of its Killing vectors is usually a question of the solution of a coupled set of partial differential equations, it is clear that that set ought, in general, to have integrability conditions. Whether they are satisfied or not is in fact a set of constraints on the connections and curvatures.

The curvature tensor of a Riemannian manifold measures the lack of commutativity of covariant derivatives, and also satisfies the first and second Bianchi identities. Considering $\widetilde{\xi}$ as some possible Killing vector, these statements are simply formulated mathematically as the three equations below:

$$\begin{aligned} \xi_{\mu;[\nu\lambda]} &= +R^\eta{}_{\mu\nu\lambda}\xi_\eta \iff \xi^\mu{}_{;[\nu\lambda]} = -R^\mu{}_{\eta\nu\lambda}\xi^\eta\ , \\ R^\eta{}_{\mu\nu\lambda} &+ R^\eta{}_{\nu\lambda\mu} + R^\eta{}_{\lambda\mu\nu} = 0\ , \\ R_{\sigma\lambda\nu\mu;\eta} &+ R_{\eta\sigma\nu\mu;\lambda} + R_{\lambda\eta\nu\mu;\sigma} = 0\ . \end{aligned} \tag{4.1}$$

We may insert into these identities the requirements for a conformal Killing vector, given on the first page, Eqs. (0.1). We begin by considering together the first two of Eqs. (4.1):

$$
\begin{aligned}
&\xi_{\mu;\nu\lambda} - \xi_{\mu;\lambda\nu} + \xi_{\nu;\lambda\mu} - \xi_{\nu;\mu\lambda} + \xi_{\lambda;\mu\nu} - \xi_{\lambda;\nu\mu} = 0 \,, \\
\Longrightarrow\; &\xi_{\mu;\nu\lambda} + \xi_{\nu;\lambda\nu} + \xi_{\lambda;\mu\nu} = \chi_{,\lambda} g_{\mu\nu} + \chi_{,\nu} g_{\mu\lambda} + \chi_{,\mu} g_{\lambda\nu} \,, \\
\Longrightarrow\; \xi_{\mu;\nu\lambda} &= -\xi_{\nu;\lambda\mu} - \xi_{\lambda;\mu\nu} + \chi_{;\lambda} g_{\mu\nu} + \chi_{;\nu} g_{\mu\lambda} + \chi_{;\mu} g_{\lambda\nu} \\
&= \xi_{\lambda;[\nu\mu]} + [-\chi_{,\mu} g_{\nu\lambda} + \chi_{,\nu} g_{\lambda\mu} + \chi_{,\lambda} g_{\mu\nu}] \\
&= R^{\eta}{}_{\lambda\nu\mu}\xi_\eta + [-\chi_{,\mu} g_{\nu\lambda} + \chi_{,\nu} g_{\lambda\mu} + \chi_{,\lambda} g_{\mu\nu}] \,,
\end{aligned} \tag{4.2}
$$

which may be thought of as the first integrability condition for our original system. However, using Eq. (2.12c), the definition of the Lie derivative for the connection, and multiplying by $g^{\mu\eta}$ to raise the first index, we may rewrite this requirement in the following more geometric form:

$$
\underset{\xi}{\mathcal{L}}\, \Gamma^{\mu}{}_{\nu\lambda} = g^{\mu\eta}[-\chi_{,\eta} g_{\nu\lambda} + \chi_{,\nu} g_{\lambda\eta} + \chi_{,\lambda} g_{\eta\nu}] \,. \tag{4.3}
$$

This is an especially nice statement relative to the geometric meaning of the difference between the cases of conformal Killing vectors, i.e., non-constant χ, and the holomorphic or true Killing vectors, where χ is a constant, possibly zero.

if χ is constant then the connection is invariant under the dragging along curves generated by that Killing vector.

We now inquire as to whether these integrability conditions have integrability conditions of their own. We do this by considering again Eqs. (4.1), applying the relation between commutators of covariant derivatives and curvature for the third covariant derivatives of our Killing vector:

$$
\xi_{\mu;\nu[\lambda\eta]} = R^{\sigma}{}_{\mu\lambda\eta}\xi_{\sigma;\nu} + R^{\sigma}{}_{\nu\lambda\eta}\xi_{\mu;\sigma} \,. \tag{4.4}
$$

We may, however, also calculate the left-hand side of this equation by taking the covariant derivative of our first integrability equations, Eq. (4.2), which gives:

$$
\xi_{\mu;\nu[\lambda\eta]} = R^{\sigma}{}_{[\lambda\nu\mu}\xi_{\sigma;\eta]} + R^{\sigma}{}_{[\lambda\nu\mu;\eta]}\xi_\sigma + [-\chi_{;\mu[\eta} g_{\nu\lambda]} + \chi_{;\nu[\eta} g_{\lambda]\mu} + \chi_{;[\lambda\eta]} g_{\mu\nu}] \,. \tag{4.5}
$$

Noting that the very last term above vanishes, because χ is a scalar, we may equate the two expressions for the commutator of the derivatives and obtain the following:

$$\begin{aligned}\xi^{\sigma}(R_{\sigma\lambda\nu\mu;\eta} - R_{\sigma\eta\nu\mu;\lambda}) + \xi^{\sigma}{}_{;\eta}R_{\sigma\lambda\nu\mu} - \xi^{\sigma}{}_{;\lambda}R_{\sigma\eta\nu\mu} + \xi^{\sigma}{}_{;\nu}R_{\sigma\mu\nu\lambda} \\ + \xi_{\mu;\sigma}R^{\sigma}{}_{\nu\eta\lambda} + \chi_{;\nu\eta}g_{\lambda\nu} - \chi_{;\nu\lambda}g_{\eta\mu} - \chi_{;\mu\eta}g_{\nu\lambda} + \chi_{;\mu\lambda}g_{\nu\eta} = 0 \ .\end{aligned} \tag{4.6}$$

If we now intervene with the second Bianchi identity, and use the definition of the Lie derivative of the curvature tensor, Eq. (8), we may rewrite the last equation in the following form:

$$\underset{\xi}{\mathcal{L}} R_{\eta\lambda\nu\mu} = 2\chi R_{\eta\lambda\nu\mu} - \chi_{;\eta[\nu}g_{\lambda\mu]} - \chi_{;\lambda[\mu}g_{\eta\nu]} \ , \tag{4.7}$$

relating the Lie derivative of the curvature tensor to the second derivatives of the conformal factor χ.

5. Specialization to Homothetic Killing Vectors

A homothetic symmetry requires that χ be constant, possibly zero of course, in which case it would actually be a *true* Killing vector. We therefore now specialize to the constant case. Specialization of the equations above gives us

$$\begin{aligned}\underset{\widetilde{\xi}}{\mathcal{L}}\, g_{\mu\nu} &= 2\chi\, g_{\mu\nu} \ , \\ \underset{\widetilde{\xi}}{\mathcal{L}}\Gamma^{\mu}{}_{\nu\lambda} =&0 = \underset{\widetilde{\xi}}{\mathcal{L}} R^{\mu}{}_{\nu\lambda\eta} \ .\end{aligned} \tag{5.1}$$

The first requirement on the second line is simply a repeat of Eqs. (4.3) in the current situation; however, it is not immediately obvious that the relation on the curvature given in that line is consistent with the previous Eq. (4.7). In order to show that consistency we consider how the raising, or lowering, of an index—via the metric—affects results for those cases where the metric is not preserved by the Lie dragging, i.e., the homothetic cases. We begin by determining the Lie derivative of the inverse metric:

$$\begin{aligned}0 = \underset{\widetilde{\xi}}{\mathcal{L}}\delta^{\mu}_{\lambda} = \underset{\widetilde{\xi}}{\mathcal{L}}\left(g^{\mu\nu}g_{\nu\lambda}\right) &= \left(\underset{\widetilde{\xi}}{\mathcal{L}}g^{\mu\nu}\right)g_{\nu\lambda} + g^{\mu\nu}\underset{\widetilde{\xi}}{\mathcal{L}}g_{\nu\lambda} \\ &= \left(\underset{\widetilde{\xi}}{\mathcal{L}}g^{\mu\nu}\right)g_{\nu\lambda} + g^{\mu\nu}2\chi\, g_{\nu\lambda} = \left(\underset{\widetilde{\xi}}{\mathcal{L}}g^{\mu\nu}\right)g_{\nu\lambda} + 2\chi\delta^{\mu}_{\lambda} \ , \\ &\Longrightarrow \underset{\widetilde{\xi}}{\mathcal{L}}g^{\mu\nu} = -2\chi\, g^{\mu\nu} \ .\end{aligned} \tag{5.2}$$

We may then use that to raise the index on the Lie derivative for the curvature:

$$
\begin{aligned}
\underset{\widetilde{\xi}}{\mathcal{L}} R^{\mu}{}_{\nu\lambda\eta} = \underset{\widetilde{\xi}}{\mathcal{L}} \left(g^{\mu\sigma} R_{\sigma\nu\lambda\eta} \right) &= \left\{ \underset{\widetilde{\xi}}{\mathcal{L}} g^{\mu\sigma} \right\} R_{\sigma\nu\lambda\eta} + g^{\mu\sigma} \left\{ \underset{\widetilde{\xi}}{\mathcal{L}} R_{\sigma\nu\lambda\eta} \right\} \\
&= -2\chi R^{\mu}{}_{\nu\lambda\eta} + 2\chi\, g^{\mu\sigma} R_{\sigma\nu\lambda\eta} = 0 \ .
\end{aligned}
\tag{5.3}
$$

Following that line of calculations, it is also then worthwhile to consider various pieces of the curvature tensor:

$$
\begin{aligned}
\underset{\widetilde{\xi}}{\mathcal{L}} \mathcal{R}_{\nu\eta} &= \underset{\widetilde{\xi}}{\mathcal{L}} R^{\mu}{}_{\nu\mu\eta} = 0 \ , \\
\underset{\widetilde{\xi}}{\mathcal{L}} \mathcal{R} &= \underset{\widetilde{\xi}}{\mathcal{L}} \left(g^{\nu\eta} \mathcal{R}_{\nu\eta} \right) = -2\chi\, g^{\nu\eta} \mathcal{R}_{\nu\eta} = -2\chi\, \mathcal{R} \ , \\
\text{and also} \quad \underset{\widetilde{\xi}}{\mathcal{L}} \Gamma_{\mu\nu\eta} &= \underset{\widetilde{\xi}}{\mathcal{L}} \left(g_{\mu\sigma} \Gamma^{\sigma}{}_{\nu\eta} \right) = 2\chi\, \Gamma_{\mu\nu\eta} \ .
\end{aligned}
\tag{5.4}
$$