Physics 570
A Detailed Example of Why/How the "Guess" Method for Determining Connections Does NOT Work for Some Metrics

I. Introduction

The basic intent is to look at some metric with a coordinate-based metric that is non-diagonal. The most interesting, physically-relevant such metric is the Kerr metric, which describes rotating central masses. To avoid complicated algebra, I will present the metric here with some functions that are not explicitly given, but simply referred to as functions of certain variables, such as $S(r,\theta)$, which, for the actual Kerr metric has the form $S(r,\theta) = \sqrt{r^2 + a^2\cos^2\theta}$. (The reader may look at the handout for the Kerr metric for the details, and for the correct forms of the connections, which were determined via the entire set of commutation coefficients.)

So, for the purposes of this example, we are considering a 4-dimensional manifold with coordinates $\{r, \theta, \varphi, t\}$ and the following non-holonomic basis for the metric:

$$\underset{\sim}{\omega}^r = R(r,\theta)\,dr\ , \quad \underset{\sim}{\omega}^\theta = S(r,\theta)\,d\theta\ , \underset{\sim}{\omega}^t = T(r,\theta)\,ddt$$

$$\underset{\sim}{\omega}^\varphi = P(r,\theta)\,d\varphi - W(r,\theta)\,dt$$

$$\mathbf{g} \equiv \eta_{\mu\nu}\,\underset{\sim}{\omega}^\mu \otimes \underset{\sim}{\omega}^\nu\ ,$$

and, of course, the choice of basis 1-forms has been made so that the quantities $\eta_{\mu\nu}$ are the usual elements of the metric tensor in special relativity, and are then constant.

To determine the connections when the components of the metric are constant, we use Cartan's First Structure Equations, shown in the first line below:

$$d\underset{\sim}{\omega}^\mu = \underset{\sim}{\omega}^\lambda \wedge \underset{\sim}{\Gamma}^\mu{}_\lambda\ ,$$

$$\underset{\sim}{\Gamma}_{\mu\lambda} + \underset{\sim}{\Gamma}_{\lambda\mu} = dg_{\mu\lambda}\ ,$$

The second line, now above, shows at least one of the reasons "why" we want to have constant metric components, since that causes the right-hand side of that line to vanish, which then informs us that the various connection 1-forms are skew-symmetric on their two indices, when they are presented with both indices on the same level—in this case, both covariant, allowing us to know that

$$\underset{\sim}{\Gamma}_{\mu\nu} = -\underset{\sim}{\Gamma}_{\nu\mu} \; ; \qquad \underset{\sim}{\Gamma}_{\mu\nu} = g_{\mu\lambda}\underset{\sim}{\Gamma}^{\lambda}{}_{\nu} \longrightarrow \eta_{\mu\lambda}\underset{\sim}{\Gamma}^{\lambda}{}_{\nu} \; ,$$

where the arrow simply reminds us that we have **chosen** a basis for 1-forms such that the metric is just the same as if we were operating totally in special relativity.

We now calculate the exterior derivatives of each of the 4 basis 1-forms:

$$\begin{aligned}
d\underset{\approx}{\omega}^r =& d\{R(r,\theta)\,dr\} = R_{,\theta} d\theta \wedge dr = \left\{\frac{R_{,\theta}}{RS}\right\}\underset{\approx}{\omega}^\theta \wedge \underset{\approx}{\omega}^r \; , \\
d\underset{\approx}{\omega}^\theta =& d\{S(r,\theta)\,d\theta\} = S_{,r} dr \wedge d\theta = \left\{\frac{S_{,r}}{SR}\right\}\underset{\approx}{\omega}^r \wedge \underset{\approx}{\omega}^\theta \; , \\
d\underset{\approx}{\omega}^t =& d\{T(r,\theta\,dt\} = T_{,r} dr \wedge dt + T_{,\theta} d\theta \wedge dt = \left\{\frac{(\log T)_{,r}}{R}\right\}\underset{\approx}{\omega}^r \wedge \underset{\approx}{\omega}^t + \left\{\frac{(\log T)_{,\theta}}{S}\right\}\underset{\approx}{\omega}^\theta \wedge \underset{\approx}{\omega}^t \; , \\
d\underset{\approx}{\omega}^\varphi =& d\{P(r,\theta)\,d\varphi - W(r,\theta)\,dt\} = P_{,r} dr \wedge d\varphi + P_{,\theta} d\theta \wedge d\varphi - W_{,r} dr \wedge dt - W_{,\theta} d\theta \wedge dt \\
&= \left\{\frac{(\log(P/W))_{,r}}{R}\right\}\underset{\approx}{\omega}^r \wedge \underset{\approx}{\omega}^\varphi - \left\{\frac{(\log(P/W))_{,\theta}}{S}\right\}\underset{\approx}{\omega}^r \wedge \underset{\approx}{\omega}^\varphi
\end{aligned}$$

where I have simplified the forms of some of the coefficients by writing them in terms of derivatives of logarithms, and the subscripts with commas indicate partial derivatives, i.e.,

$$R_{,\theta} \equiv \frac{\partial R(r,\theta)}{\partial\theta} \; .$$

We then write out the First Structure Equations for each of these, remembering the form of the metric and the skew-symmetry of the connection 1-forms on their labels, where you should ensure you understand the various signs below:

$$\begin{aligned}
d\underset{\approx}{\omega}^r &= \underset{\approx}{\omega}^\nu \wedge \underset{\sim}{\Gamma}_{r\nu} = \underset{\approx}{\omega}^\theta \wedge \underset{\sim}{\Gamma}_{r\theta} + \underset{\approx}{\omega}^\varphi \wedge \underset{\sim}{\Gamma}_{r\varphi} + \underset{\approx}{\omega}^t \wedge \underset{\sim}{\Gamma}_{rt} \; , \\
d\underset{\approx}{\omega}^\theta &= \underset{\approx}{\omega}^\nu \wedge \underset{\sim}{\Gamma}_{\theta\nu} = -\underset{\approx}{\omega}^{\vec{r}} \wedge \underset{\sim}{\Gamma}_{r\theta} + \underset{\approx}{\omega}^\varphi \wedge \underset{\sim}{\Gamma}_{\theta\varphi} + \underset{\approx}{\omega}^t \wedge \underset{\sim}{\Gamma}_{\theta t} \; , \\
d\underset{\approx}{\omega}^\varphi &= \underset{\approx}{\omega}^\nu \wedge \underset{\sim}{\Gamma}_{\varphi\nu} = -\underset{\approx}{\omega}^r \wedge \underset{\sim}{\Gamma}_{r\varphi} - \underset{\approx}{\omega}^\theta \wedge \underset{\sim}{\Gamma}_{\theta\varphi} + \underset{\approx}{\omega}^t \wedge \underset{\sim}{\Gamma}_{\varphi t} \; , \\
d\underset{\approx}{\omega}^t &= -\underset{\approx}{\omega}^\nu \wedge \underset{\sim}{\Gamma}_{t\nu} = \underset{\approx}{\omega}^r \wedge \underset{\sim}{\Gamma}_{rt} + \underset{\approx}{\omega}^\theta \wedge \underset{\sim}{\Gamma}_{\theta t} + \underset{\approx}{\omega}^\varphi \wedge \underset{\sim}{\Gamma}_{\varphi t} \; .
\end{aligned}$$

At this point we can specify the details of the Assumption involved in the Guess method for the determination of the various connection 1-forms from this data: In the two sets of 4 equations above we have two different presentations of the exterior derivatives of the various basis 1-forms. The idea is to consider each pair of the equalities for any one of the basis 1-forms, **and to assume** that the 3 individual terms that present a given $d\underset{\sim}{\omega}^{\mu}$ in terms of connections are "independent" of each other, by which I mean, even more carefully that the 3 terms do not have parts which cancel each other in their sum. Now, let's try to understand this even better via the explicit forms we have. We begin with the pair of expressions for $d\underset{\sim}{\omega}^{r}$:

$$\left\{\frac{(\log R)_{,\theta}}{S}\right\}\underset{\sim}{\omega}^{\theta}\wedge\underset{\sim}{\omega}^{r} = d\underset{\sim}{\omega}^{r} = \underset{\sim}{\omega}^{\nu}\wedge\underset{\sim}{\Gamma}_{r\nu} = \underset{\sim}{\omega}^{\theta}\wedge\underset{\sim}{\Gamma}_{r\theta} + \underset{\sim}{\omega}^{\varphi}\wedge\underset{\sim}{\Gamma}_{r\varphi} + \underset{\sim}{\omega}^{t}\wedge\underset{\sim}{\Gamma}_{rt}$$

In order for this equality to be true, one of the terms on the right-hand side must be proportional to to $\underset{\sim}{\omega}^{\theta}\wedge\underset{\sim}{\omega}^{r}$, since this is the only term on the left-hand side. Looking at the terms on the right-hand side we see that only one of those terms—the first one—has that possibility. As well, we have assumed that the 3 individual terms on the right-hand side do not cancel any parts of a different one; therefore, those other two terms must vanish, since there are no more terms on the left-hand side that could accommodate them. This leads us to the following partial knowledge concerning those connection 1-forms involved in this equation:

$$\underset{\sim}{\Gamma}_{r\theta} = \left\{\frac{(\log R)_{,\theta}}{S}\right\}\underset{\sim}{\omega}^{r}\ , \qquad \underset{\sim}{\Gamma}_{r\varphi}\propto\underset{\sim}{\omega}^{\varphi}\ , \quad \underset{\sim}{\Gamma}_{rt}\propto\underset{\sim}{\omega}^{t}\ ,$$

where the two proportionality statements are what is necessary if those terms are, **individually**, to vanish.

We now continue with this same sort of equality generated by the two forms for $d\underset{\sim}{\omega}^{\theta}$:

$$\left\{\frac{S_{,r}}{SR}\right\}\underset{\sim}{\omega}^{r}\wedge\underset{\sim}{\omega}^{\theta} = d\underset{\sim}{\omega}^{\theta} == -\underset{\sim}{\omega}^{\vec{r}}\wedge\underset{\sim}{\Gamma}_{r\theta} + \underset{\sim}{\omega}^{\varphi}\wedge\underset{\sim}{\Gamma}_{\theta\varphi} + \underset{\sim}{\omega}^{t}\wedge\underset{\sim}{\Gamma}_{\theta t}\ .$$

Comparing the two sides, and invoking our assumption called the guess method, this equality gives us the following new pieces of information:

$$-\underset{\sim}{\Gamma}_{\theta r} = \underset{\sim}{\Gamma}_{r\theta} = \left\{\frac{(\log S)_{,r}}{R}\right\}\underset{\sim}{\omega}^{\theta}\ , \quad \underset{\sim}{\Gamma}_{\theta\varphi}\propto\underset{\sim}{\omega}^{\varphi}\ , \quad \underset{\sim}{\Gamma}_{\theta t}\propto\underset{\sim}{\omega}^{t}\ .$$

At this point, we see that we have two different presentations for $\underset{\sim}{\Gamma}_{r\theta}$, in two different directions. This would be possible **if and only if** those scalars—the components in the two different directions—were both zero. In our case, of the Kerr metric, they are in fact moderately complicated functions of both r and θ, and are definitely different from zero.

This is of course most unfortunate, and unacceptable. We have reached a contradiction, which tells us that our assumption was invalid! The guess method did not work for this particular metric! We can stop at this point, although, in principle there would be two more equations to compare, those for $d\underset{\sim}{\omega}^{\varphi}$ and for $d\underset{\sim}{\omega}^{t}$; however, the method has already failed, so there is no good reason to continue with it.

From the point of view of continuing to explain the method, I will point out that we can see that we were accumulating proportionalities, without known coefficients, for various of the connection 1-forms. In the event that the method had not led to a contradiction, it might have occurred that we found that some particular one of the connection 1-forms was required to be proportional to two distinct basis vectors, **but** without, in either case, giving us an explicit form for the coefficient. In that case, consistency could still be acquired by insisting that this particular connection 1-form was simply zero.

For this metric—which has the Kerr metric as a special case, with particular functions for each of the arbitrary functions in our basis for 1-forms—the evaluation of the 1-forms will have to retreat to the formula we have which gives the components of the connections in terms of the commutation coefficients, which would then have to first be determined.
For the actual Kerr metric, one can see those results in the appropriate handout. Looking there it is straightforward to observe that each of the 6 independent 1-forms, that constitute the set referred to as "the connection 1-forms," has a non-zero component associated with two different basis 1-forms, thereby ensuring that the "guess" method would not work.