

## The Kerr Metric—for a Rotating Black Hole

In Boyer-Lindquist coordinates the Kerr metric may be written in the following form,

$$\begin{aligned}
 ds^2 &= \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + (r^2 + a^2) \sin^2 \theta d\varphi^2 - dt^2 + \frac{2mr}{\Sigma} (a \sin^2 \theta d\varphi - dt)^2, \\
 &= \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{A}{\Sigma} \sin^2 \theta d\varphi^2 - 2 \frac{2mar}{\Sigma} \sin^2 \theta d\varphi dt - \left( 1 - \frac{2mr}{\Sigma} \right) dt^2, \\
 &\equiv g_{\mu\nu}(r, \theta) dr^\mu dr^\nu,
 \end{aligned} \tag{1}$$

which describes the gravitational field exterior to a rotating black hole of mass  $m$  and angular momentum per unit mass of amount  $a$ , pointing along the positive  $\hat{z}$ -direction. I also note that the determinant of the metric satisfies  $g = -\Sigma^2 \sin^2 \theta$ .

It is reasonable to describe this system via a “locally, non-rotating,” (orthonormal) tetrad, i.e., a LNRF—also called a ZAMO, because this observer has zero angular momentum—of the following form:

$$\left. \begin{aligned}
 \varpi^r &= \sqrt{\frac{\Sigma}{\Delta}} dr, & \varpi^\theta &= \sqrt{\Sigma} d\theta, & \varpi^t &= \sqrt{\frac{\Sigma\Delta}{A}} dt, \\
 \varpi^\varphi &= \sqrt{\frac{A}{\Sigma}} \sin \theta d\varphi - \frac{2mar \sin \theta}{\sqrt{\Sigma A}} dt = \sqrt{\frac{A}{\Sigma}} \sin \theta (d\varphi - \omega dt),
 \end{aligned} \right\} \varpi^{\hat{\mu}} \equiv Y^{\hat{\mu}}{}_{\alpha} dx^{\alpha}$$

$$\text{with } \left\{ \begin{aligned}
 \Sigma &\equiv r^2 + (a \cos \theta)^2, \\
 \Delta &\equiv r^2 + a^2 - 2mr, & \omega &\equiv \frac{2mar}{A} = -\frac{g^{\varphi t}}{g^{tt}} = -\frac{g_{\varphi t}}{g_{\varphi\varphi}}, \\
 A &\equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta = (r^2 + a^2) \Sigma + 2ma^2 r \sin^2 \theta = \Delta \Sigma + 2mr(r^2 + a^2).
 \end{aligned} \right. \tag{2}$$

On the other hand the associated, reciprocal basis of tangent vectors is

$$\left. \begin{aligned}
 \tilde{e}_r &= \sqrt{\frac{\Delta}{\Sigma}} \partial_r, & \tilde{e}_\theta &= \frac{1}{\sqrt{\Sigma}} \partial_\theta, & \tilde{e}_\varphi &= \sqrt{\frac{\Sigma}{A}} \frac{1}{\sin \theta} \partial_\varphi, \\
 \tilde{e}_t &= \sqrt{\frac{A}{\Sigma\Delta}} (\partial_t + \omega \partial_\varphi) = \sqrt{\frac{A}{\Sigma\Delta}} \partial_t + \frac{2mar}{\sqrt{A\Sigma\Delta}} \partial_\varphi,
 \end{aligned} \right\} \tilde{e}_{\hat{\lambda}} \equiv X^{\beta}{}_{\hat{\lambda}} \partial_{x^\beta}, \tag{3}$$

where the matrices  $X^{\beta}{}_{\hat{\lambda}}$  and  $Y^{\hat{\mu}}{}_{\alpha}$  are inverse to each other. Moreover, it follows that for some arbitrary 4-vector,  $\tilde{\ell}$ , and its associated 1-form,  $\ell$ , we have the following relations

$$\tilde{\ell} = \ell^\alpha \partial_{x^\alpha} = \ell^{\hat{\mu}} \tilde{e}_{\hat{\mu}}, \quad \ell = \ell_\beta dx^\beta = \ell_{\hat{\nu}} \varpi^{\hat{\nu}}, \quad \ell^{\hat{\lambda}} = Y^{\hat{\lambda}}{}_{\alpha} \ell^\alpha, \quad \ell_{\hat{\mu}} = X^{\beta}{}_{\hat{\mu}} \ell_\beta.$$

where it is important to be careful as to when the summations are performed on the row, or the column, index of the matrices.

After some considerable calculation, one finds that, relative to this choice of LNRF, it is sufficient to have 8 functions, of  $r$  and  $\theta$ , as well as  $m$  and  $a$ , in order to describe the connections. We find that

$$\begin{pmatrix} \tilde{\Gamma}_{r\theta} \\ \tilde{\Gamma}_{r\varphi} \\ \tilde{\Gamma}_{rt} \\ \tilde{\Gamma}_{\theta\varphi} \\ \tilde{\Gamma}_{\theta t} \\ \tilde{\Gamma}_{\varphi t} \end{pmatrix} = \begin{pmatrix} B & C & 0 & 0 \\ 0 & 0 & D & -F \\ 0 & 0 & -F & \frac{r-m}{\sqrt{\Sigma\Delta}} + D \\ 0 & 0 & -\frac{\cot\theta}{\sqrt{\Sigma}} - G & -H \\ 0 & 0 & -H & -G \\ -F & -H & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\omega}^r \\ \tilde{\omega}^\theta \\ \tilde{\omega}^\varphi \\ \tilde{\omega}^t \end{pmatrix}$$

where

$$\begin{cases} B \equiv \frac{1}{\sqrt{\Sigma}} \partial_\theta (\log \sqrt{\Sigma}), & C \equiv -\sqrt{\frac{\Delta}{\Sigma}} \partial_r (\log \sqrt{\Sigma}), \\ D \equiv -\sqrt{\frac{\Delta}{\Sigma}} \partial_r (\log \sqrt{A/\Sigma}), & \frac{r-m}{\sqrt{\Sigma\Delta}} + D = \sqrt{\frac{\Delta}{\Sigma}} \partial_r (\log \sqrt{\Sigma\Delta/A}), \\ G \equiv -\frac{1}{\sqrt{\Sigma}} \partial_\theta (\log \sqrt{\Sigma/A}), & \frac{\cot\theta}{\sqrt{\Sigma}} + G = \frac{1}{\sqrt{\Sigma}} \partial_\theta (\log(\sqrt{A/\Sigma} \sin \theta)), \\ F \equiv -\frac{A \sin \theta}{2\Sigma^{3/2}} \partial_r \omega, & H = -\frac{A \sin \theta}{2\Delta^{1/2}\Sigma^{3/2}} \partial_\theta \omega. \end{cases} \quad (4)$$

We may then go further, yet, and describe the curvature itself in terms of only 4 distinct functions:

$$\begin{aligned} R_{r\theta r\theta} &= -R_{t\varphi t\varphi} = -Q_1, & R_{r\theta\varphi t} &= +Q_2, \\ R_{rt r t} &= -R_{\theta\varphi\theta\varphi} = -Q_1 \frac{2+z}{1-z}, & R_{rt\theta t} &= R_{r\varphi\theta\varphi} = S Q_2, \\ R_{rt r \varphi} &= -R_{\theta t\theta\varphi} = S Q_1, & R_{rt\theta\varphi} &= -Q_2 \frac{2+z}{1-z}, \\ R_{\theta t\theta t} &= -R_{r\varphi r\varphi} = Q_1 \frac{1+2z}{1-z}, & R_{r\varphi\theta t} &= -Q_2 \frac{1+2z}{1-z}. \end{aligned} \quad (5)$$

$$\text{with } \begin{cases} Q_1 \equiv mr(r^2 - 3a^2 \cos^2 \theta)/\Sigma^3, & Q_2 \equiv ma \cos \theta (3r^2 - a^2 \cos^2 \theta)/\Sigma^3, \\ S \equiv 3a \sin \theta \sqrt{\Delta} (r^2 + a^2)/A, & z \equiv \Delta \left( \frac{a \sin \theta}{r^2 + a^2} \right)^2. \end{cases}$$

## Geodesic Trajectories:

If  $\tilde{u}$  is the tangent vector field for a geodesic curve, then it may be related to the proper-time derivative of the coordinates and to the standard notions for 3-velocity and  $\gamma$  factor as follows:

$$\gamma \equiv u^{\hat{t}} = \sqrt{\frac{\Sigma \Delta}{A}} \frac{dt}{d\tau}, \quad v^i \equiv \frac{u^{\hat{i}}}{u^{\hat{t}}} \implies \begin{pmatrix} \frac{\sqrt{A}}{\Delta} \frac{dr}{d\tau} \\ \sqrt{\frac{A}{\Delta}} \frac{d\theta}{d\tau} \\ \frac{A \sin \theta}{\Sigma \sqrt{\Delta}} \left( \frac{d\varphi}{d\tau} - \omega \right) \end{pmatrix}. \quad (6)$$

then the coordinates that describe that trajectory satisfy the following equations:

$$\begin{aligned} \Sigma \frac{dr}{d\tau} &= \pm \sqrt{V_r} \equiv \pm \sqrt{T^2 - \Delta[(\mu r)^2 + (L - aE)^2 + Q]}, \\ \Sigma \frac{d\theta}{d\tau} &= \pm \sqrt{V_\theta} \equiv \pm \sqrt{Q - [a^2(\mu^2 - E^2) + L^2/(\sin^2 \theta)] \cos^2 \theta}, \\ \Sigma \frac{d\varphi}{d\tau} &= -[aE - L/(\sin^2 \theta)] + aT/\Delta, \\ \Sigma \frac{dt}{d\tau} &= -a(aE \sin^2 \theta - L) + (r^2 + a^2)T/\Delta, \end{aligned} \quad (7)$$

$$T \equiv E(r^2 + a^2) - La, \quad E = -p_t, \quad L = p_\varphi,$$

$$Q = p_\theta^2 + [a^2(\mu^2 - p_t^2) + p_\varphi^2/(\sin^2 \theta)] \cos^2 \theta.$$

Here the quantities  $E$ ,  $L$ ,  $Q$ , and  $\mu = 1$  or  $0$  are constants of the motion; the first two are caused by the obvious Killing vectors,  $\partial_t$  and  $\partial_\varphi$  while the third is caused by a non-obvious Killing tensor and the fourth is simply the value that describes if the geodesic is timelike or null.

## Interesting Surfaces

There is an outer, and an inner, horizon, where  $g_{rr}$  becomes infinite, at

$$r_\pm = m \pm \sqrt{m^2 - a^2}. \quad (8)$$

Between these two horizons  $\partial_r$  is timelike. However, there is also a surface of static limit, at which  $g_{tt}$  vanishes, so that  $\partial_t$  becomes null at that point, and spacelike between these two surfaces. The volume between this surface and the outer horizon is referred to as the *ergosphere*.

Within the ergosphere, there are no timelike curves that do not rotate along with the star. The inner horizon is a Cauchy horizon, and within it,  $\partial_r$  again becomes spacelike. Lastly, there is a true curvature singularity, at the boundary of the disc defined by  $\Sigma \equiv r^2 + a^2 \cos^2 \theta = 0$ . With proper acceleration one can avoid this singularity on a timelike path.