General Lorentz Boost Transformations,
Acting on Some Important Physical Quantities

We are interested in transforming measurements made in a reference frame $\mathcal{O}'$ into measurements of the same quantities as made in a reference frame $\mathcal{O}$, where the reference frame $\mathcal{O}$ measures $\mathcal{O}'$ to be moving with constant velocity $\vec{v}$, in an arbitrary direction, which then associates with that velocity the scale factor $\gamma \equiv \gamma_v \equiv 1/\sqrt{1-(v/c)^2}$. It therefore follows that $\mathcal{O}'$ measures the reference frame $\mathcal{O}$ to be moving with velocity $-\vec{v}$, so that all formulae discussed below may be re-formulated by switching quantities with a prime, i.e., measurements made in the frame $\mathcal{O}'$, with quantities without a prime, while changing the sign of the velocity.
It is useful to divide all other vector quantities into their parts which are parallel and perpendicular to $\vec{v}$; for instance, for the location vector $\vec{r}$ we write

$$\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp} = (\hat{v}\cdot\vec{r})\hat{v} + \hat{v}\times(\vec{r}\times\hat{v}) \; , \tag{1}$$

where the first term, after the last equals sign, is that portion of $\vec{r}$ which is in the same direction as $\vec{v}$, while the second term is the remainder of $\vec{r}$, which is of course perpendicular to $\vec{v}$.

The appropriate Lorentz transformation equations for the location vector are then

$$\vec{r}_{\parallel} = \gamma[\vec{r}\,'_{\parallel} + t'\vec{v}] \; , \qquad \vec{r}_{\perp} = \vec{r}\,'_{\perp} \; , \tag{2a}$$

$$t = \gamma[t' + \vec{v}\cdot\vec{r}\,'/c^2] \; , \tag{2b}$$

$$\text{or} \quad \vec{r} = \vec{r}\,' + (\gamma-1)(\vec{r}\,'\cdot\hat{v})\hat{v} + \gamma t'\vec{v} \; . \tag{3}$$

Since the transformations mix together $\vec{r}$ and t, it is profitable to devise a method to describe the quantities so that the transformation is a linear one, that can be considered using matrices. To do this we create a 4-dimensional vector, usually referred to simply as a "4-vector," which then transforms between the two frames via a so-called *"boost matrix"*, denoted by $\Lambda(\vec{v})$. We denote the location-and-time 4-vector by $\tilde{r}$, and write the earlier transformations in the following way, with a 4×4 matrix performing the transformation, where we must note that the superscript ^{T} means the transpose of the matrix. It is somewhat unfortunate, however, that the matrix is quite messy for the general case; therefore, often the parallel and perpendicular decompositions above are much simpler to use, although of course this is not easy when there are more than two reference frames to consider:

$$\begin{pmatrix} \vec{r} \\ ct \end{pmatrix} \equiv \widetilde{r} = \Lambda(\vec{v})\widetilde{r}\,' = \Lambda(\vec{v}) \begin{pmatrix} \vec{r}\,' \\ ct' \end{pmatrix} ,$$

$$\Lambda(\vec{v}) = \begin{pmatrix} I_3 + (\gamma - 1)\hat{v}\hat{v}^T & \gamma\vec{v}/c \\ \gamma\vec{v}^T/c & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 1 + (\gamma - 1)(\hat{v})_x(\hat{v})_x & (\gamma - 1)(\hat{v})_x(\hat{v})_y & (\gamma - 1)(\hat{v})_x(\hat{v})_z & \gamma\, v_x/c \\ (\gamma - 1)(\hat{v})_x(\hat{v})_y & 1 + (\gamma - 1)(\hat{v})_y(\hat{v})_y & (\gamma - 1)(\hat{v})_y(\hat{v})_z & \gamma\, v_y/c \\ (\gamma - 1)(\hat{v})_x(\hat{v})_z & (\gamma - 1)(\hat{v})_y(\hat{v})_z & 1 + (\gamma - 1)(\hat{v})_z(\hat{v})_z & \gamma\, v_z/c \\ \gamma\, v_x/c & \gamma\, v_y/c & \gamma\, v_z/c & \gamma \end{pmatrix} \tag{4}$$

Now that we have this structure, however, we may create other 4-vectors, all of which transform in the same way, and may also use those transformations to determine the relation to the transformations of the ordinary 3-vector quantities. I will outline several of these below; however, it is probably useful here to just re-write the above (rather messy-looking) matrix as it would appear acting on any arbitrary 4-vector, which we designate by the symbol $\widetilde{p}$, with its 3-dimensional part denoted by $\vec{p}$ and its 4-th component denoted by p^4. We have

$$\begin{pmatrix} \vec{p} \\ p^4 \end{pmatrix} \equiv \widetilde{p} = \Lambda(\vec{v})\widetilde{p}\,' = \begin{pmatrix} \vec{p}\,' + (\gamma - 1)(\hat{v}\cdot\vec{p}\,')\hat{v} + \gamma\,(p^4)'\vec{v}/c \\ \gamma[(\vec{v}\cdot\vec{p}\,')/c + (p^4)'] \end{pmatrix} \tag{5}$$

Arbitrary 4-vectors have associated with them an invariant quantity which is a generalization of "length" or distance in 3-dimensional space, which we will usually refer to as simply *the square of the 4-vector,* even though it may be positive, negative, or even zero. Taking this arbitrary 4-vector $\widetilde{p}$, we have

$$\widetilde{p}^2 \equiv \widetilde{p}\cdot\widetilde{p} \equiv \vec{p}^{\,2} - (p^4)^2 = (\vec{p}\,')^2 - [(p^4)']^2 = (\widetilde{p}\,')^2 , \tag{6}$$

which has a value that is independent of the observer, i.e., which is invariant under Lorentz transformations.

There are also other, important, physical quantities that are not part of 4-vectors, but, rather, something more complicated. The most immediate ones are the electromagnetic fields, which, in (4-dimensional) spacetime belong as the 6 components of a second-rank, antisymmetric tensor. For easy reference I will also describe them in these notes, after the discussion of 4-vectors and their associated 3-vectors.

A very important quantity involved with the definitions used to describe the behavior of any particle, at location $\vec{r}(t)$, is the so-called proper time, τ, which is a scalar quantity. We know that any two points on the world line of some observer are "time-like separated." Therefore, for any two such points, A and B, we define the difference of their proper time by

$$(\Delta\tau)^2_{AB} \equiv (\Delta t)^2_{AB} - (\Delta\vec{r})^2_{AB}/c^2 \ , \tag{7}$$

where the sign is chosen so that $\Delta\tau_{AB} \equiv \tau_B - \tau_A$ is positive when B is to the future of A, along the worldline in question. **This is just the time shown on the "wrist-watch" of the observer whose worldline this is.** This quantity is an invariant one, having the same value no matter which observer happens to measure it, i.e., happens to be making measurements on this particular worldline's behavior. As well, we see that one may label uniquely points on a given worldline by their value of τ, relative to some chosen origin and some chosen scale of units, i.e., seconds, years, etc. Therefore we may take derivatives along that worldline with respect to the proper time, τ of a particle that has location $\vec{r}(t)$:

1. The velocity 4-vector is denoted by $\widetilde{u}$—to distinguish it from the velocity between reference frames denoted by $\vec{v}$—and has its transformation law given by

$$\frac{d}{d\tau}\widetilde{r} \equiv \widetilde{u} = \gamma_u \begin{pmatrix} \vec{u} \\ c \end{pmatrix} \ , \quad \text{where} \quad \vec{u} \equiv \frac{d}{dt}\vec{r} \ , \quad \frac{dt}{d\tau} = \gamma_u \ . \tag{8}$$

Note that the 4-dimensional square of the 4-velocity is given by the following constant value:

$$(\widetilde{u})^2 \equiv \widetilde{u}\cdot\widetilde{u} = \gamma_u^2[\vec{u}^2 - c^2] = -c^2 \ . \tag{9}$$

2. For a particle of mass m, moving with velocity $\vec{u}$, the energy-momentum 4-vector is given by

$$m\widetilde{u} \equiv \widetilde{p} \equiv \begin{pmatrix} \vec{p} \\ E/c \end{pmatrix} = m\gamma_u \begin{pmatrix} \vec{u} \\ c \end{pmatrix} \ .$$

Note that the mass m here is also sometimes referred to as *the rest mass* of the particle, since it is to be measured in a reference frame where the particle is at rest, which is therefore an invariant quantity:

$$-\widetilde{p}^{\,2}/c^2 = -[m\widetilde{u}]^2/c^2 = -m^2(-c^2)/c^2 = m^2 \ . \tag{10}$$

3. The acceleration 4-vector is then given by

$$\widetilde{a} \equiv \frac{d}{d\tau}\widetilde{u} = \frac{d^2}{d\tau^2}\widetilde{r} = \gamma_u^2 \begin{pmatrix} \vec{a} + \gamma_u^2(\vec{u}\cdot\vec{a})\vec{u}/c^2 \\ \gamma_u^2(\vec{u}\cdot\vec{a})/c \end{pmatrix} \ , \quad \text{where} \quad \vec{a} \equiv \frac{d}{dt}\vec{u} = \frac{d^2}{dt^2}\vec{r} \,. \tag{11}$$

4. The 4-vector force is defined so that (the appropriate generalization of) Newton's Second Law is still true, where we also recall that $dE/dt = \vec{F}\cdot\vec{u}$, with $\vec{F}$ being the usual 3-dimensional force vector:

$$\widetilde{K} \equiv \frac{d}{d\tau}\widetilde{p}\,, \qquad \widetilde{K} = \gamma_u \begin{pmatrix} \vec{F} \\ \frac{1}{c}dE/dt \end{pmatrix}\,, \tag{12}$$

Using these definitions, and the fact that each of them is a 4-vector and therefore transforms very simply by multiplication by $\Lambda(\vec{v})$, we may work out the Lorentz transformations of the associated 3-vectors, which are, in general, as expected, not very nice, except for the 3-momentum and energy/c, which transform exactly the same way as does the 3-location and c(time):

1. The 3-velocity, $\vec{u}$, and its associated function γ_u:

$$\vec{u}_\parallel = \frac{\vec{u}'_\parallel + \vec{v}}{1+\vec{v}\cdot\vec{u}'/c^2}\,,\ \vec{u}_\perp = \frac{\gamma_v^{-1}\,\vec{u}'_\perp}{1+\vec{v}\cdot\vec{u}'/c^2}\,, \tag{13a}$$

$$\text{or}\quad \vec{u} = \frac{\gamma_v^{-1}\vec{u}' + (1-\gamma_v^{-1})(\hat{v}\cdot\vec{u}')\hat{v} + \vec{v}}{1+\vec{v}\cdot\vec{u}'/c^2}\,, \tag{13b}$$

$$\text{and}\quad \frac{1}{\sqrt{1-(u/c)^2}} \equiv \gamma_u = \gamma_v(1+\vec{v}\cdot\vec{u}'/c^2)\gamma_{u'} = \gamma_v\frac{(1+\vec{v}\cdot\vec{u}'/c^2)}{\sqrt{1-(u'/c)^2}}\,,\ . \tag{14}$$

2. The 3-momentum, $\vec{p}$ and the energy E:

$$\vec{p}_\parallel = \gamma_v(\vec{p}'_\parallel + E'\vec{v}/c)\,,\quad \vec{p}_\perp = \vec{p}'_\perp\,,\quad E = \gamma_v(E' + \vec{v}\cdot\vec{p}')\,, \tag{15a}$$

$$\text{or}\quad \vec{p} = \vec{p}' + (\gamma_v - 1)(\vec{p}'\cdot\hat{v})\hat{v} + \gamma_v E'\vec{v}/c^2\,. \tag{15b}$$

3. The 3-acceleration, $\vec{a}$:

$$\begin{aligned}\vec{a}_\parallel = \frac{\gamma_v^{-3}}{(1+\vec{v}\cdot\vec{u}'/c^2)^3}\,\vec{a}'_\parallel\,,\quad \vec{a}_\perp &= \frac{\gamma_v^{-2}}{(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a}'_\perp + \vec{v}\times(\vec{a}'\times\vec{u}')/c^2\right\}\\ &= \frac{1-(v/c)^2}{(1+\vec{v}\cdot\vec{u}'/c^2)^2}\left\{\vec{a}'_\perp - \frac{\vec{v}\cdot\vec{a}'/c}{1+\vec{v}\cdot\vec{u}'/c^2}\frac{\vec{u}'_\perp}{c}\right\}\\ \text{or}\quad \vec{a} = \frac{\gamma_v^{-3}}{(1+\vec{v}\cdot\vec{u}'/c^2)^3}&\left\{\vec{a}'_\parallel + \gamma_v\,\vec{a}'_\perp + \gamma_v\,\vec{v}\times(\vec{a}'\times\vec{u}')/c^2\right\}\,.\end{aligned} \tag{16}$$

4. The 3-force, and its associated quantity $\dot{E} = dE/dt = \vec{F}\cdot\vec{u}$, when it is acting on an object with velocity $\vec{u}$:

$$\vec{F}_\parallel = \frac{\vec{F}'_\parallel + (\vec{u}'\cdot\vec{F}')\vec{v}/c^2}{1+\vec{v}\cdot\vec{u}'/c^2}\,,\qquad \vec{F}_\perp = \frac{\gamma_v^{-1}\vec{F}'_\perp}{1+\vec{v}\cdot\vec{u}'/c^2}\,, \tag{17a}$$

$$\text{or}\quad \vec{F} = \frac{\gamma_v^{-1}\vec{F}' + (1-\gamma_v^{-1})(\hat{v}\cdot\vec{F}')\,\hat{v} + (\vec{u}'\cdot\vec{F}')\vec{v}/c^2}{1+\vec{v}\cdot\vec{u}'/c^2}\,, \tag{17b}$$

and a related set of equations, showing the true relationships between the 3-force and the 3-acceleration when objects are moving quite fast:

$$\vec{F} = \gamma_u m[\vec{a} + \gamma_u^2 (\vec{u}\cdot\vec{a})\vec{u}/c^2] \ , \tag{18a}$$

$$\text{or} \quad \vec{F}_{\parallel\vec{u}} = \gamma_u^3 \, m \, \vec{a}_{\parallel\vec{u}} \ , \quad \vec{F}_{\perp\vec{u}} = \gamma_u \, m \, \vec{a}_{\perp\vec{u}} \ . \tag{18b}$$

Notice that the factor involving γ_u that multiplies the Newtonian-like quantity $m\,\vec{a}$ is quite different for the 2 cases of acceleration parallel to the frame velocity $\vec{v}$ and perpendicular to it! This makes it impossible to create some sort of "new" definition of mass which would allow the Newtonian relation between $\vec{F}$ and $m\vec{a}$.

We now want to say a little bit about the electromagnetic fields. They properly belong as components of a particular second-rank, antisymmetric tensor, usually referred to as the Faraday. Choosing the simplest, Cartesian basis for our 3-vectors, the Faraday is presented as the following matrix:

$$F^{\mu\nu} = \begin{pmatrix} 0 & B^z & -B^y & -E^x/c \\ -B^z & 0 & B^x & -E^y/c \\ B^y & -B^x & 0 & -E^z/c \\ E^x/c & E^y/c & E^z/c & 0 \end{pmatrix} \ . \tag{19}$$

Since we know that a 4-vector transforms via the Lorentz boost matrix, as described earlier, via $\widetilde{r} = \Lambda(\vec{v})\widetilde{r}\,'$, we may surmise, or believe, that this 2-index object should transform as

$$F^{\mu\nu} = \Lambda(\vec{v})^\mu{}_\alpha F'^{\alpha\beta} \Lambda(\vec{v})^\nu{}_\beta \qquad \Longleftrightarrow \qquad F = \Lambda(\vec{v})\, F' \Lambda(\vec{v})^T \ , \tag{20a}$$

where the second equality is simply the same as the first one, but written in terms of square matrices, using the usual rules for matrix arithmetic to express the pair of (implied) sums in the first formulation.

As before, such a set of transformation equations is perhaps easier to understand if written out in terms of those components parallel and perpendicular to the transformation velocity:

$$\left.\begin{aligned} \vec{E}_\parallel &= \vec{E}\,'_\parallel \ , \\ \vec{E}_\perp &= \gamma_v\{\vec{E}\,'_\perp - \vec{v}\times\vec{B}\,'\} \ , \end{aligned}\right\} \qquad \left\{\begin{aligned} \vec{B}_\parallel &= \vec{B}\,'_\parallel \ , \\ \vec{B}_\perp &= \gamma_v\{\vec{B}\,'_\perp + \vec{v}\times\vec{E}\,'/c^2\} \ , \end{aligned}\right. \tag{20b}$$

Notice that these equations are really rather different from those for a 4-vector, especially in the sense that for the 3-vector portion of a 4-vector it is that part perpendicular to $\vec{v}$ which is unchanged, while for the 3-vectors $\vec{E}$ and $\vec{B}$, it is their portion parallel to the velocity which is unchanged.

Again, as a useful reference, there is a second way to present the elements of the electromagnetic field, in terms of a different, second-rank, antisymmetric tensor, referred to as *the dual of the Faraday*, denoted by ${}^*F$, and also sometimes named *the Maxwell tensor.* It is presented, in the same way as above by the following matrix:

$$({}^*F)^{\mu\nu} = \begin{pmatrix} 0 & -E^z/c & +E^y/c & -B^x \\ +E^z/c & 0 & -E^x/c & -B^y \\ -E^y/c & +E^x/c & 0 & -B^z \\ +B^x & +B^y & +B^z & 0 \end{pmatrix} . \tag{21}$$

In order to understand how these two presentations are related see notes on the Levi-Civita weighted tensor, and its use to divide skew-symmetric matrices into two distinct parts. Of course the individual transformation equations generated by making a Lorentz transform of the Maxwell are the same as those generated by using the Faraday, as given in Eqs. (20b).