Exact solution to the averaging problem in cosmology

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The exact solution of a two–scale Buchert average of the Einstein equations is derived for an inhomogeneous universe which represents a close approximation to the observed universe. The two scales represent voids, and the bubble walls surrounding them within which clusters of galaxies are located. As described elsewhere [New J. Phys. **9** (2007) 377], apparent cosmic acceleration can be recognised as a consequence of quasilocal gravitational energy gradients between observers in bound systems and the volume average position in freely expanding space. With this interpretation, the new solution presented here replaces the Friedmann solutions, in representing the average evolution of a matter–dominated universe without exotic dark energy, while being observationally viable.

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At the time of last–scattering the distribution of matter in the universe was very smooth, given the evidence of the cosmic microwave background (CMB). At the present epoch, by contrast, the universe is very lumpy on scales less than 100–300Mpc, with clusters of galaxies strung in filaments and bubbles surrounding huge voids. Some 40–50% of the present volume [1] of the universe is in voids of order $30h^{-1}$Mpc in diameter, h being the dimensionless Hubble parameter, $H_0 = 100h\,\mathrm{km\ sec^{-1}\,Mpc^{-1}}$, and when smaller and larger voids [2] are taken into account, then the observable universe is "void–dominated".

In spite of present–day inhomogeneity, a broadly isotropic Hubble flow is observed when one averages on sufficiently large scales. This is taken as justification for assuming that cosmic evolution can be modelled by the Friedmann equation for a smooth fluid, despite the obvious observational evidence that galaxies are not smoothly distributed. To achieve agreement with a number of independent observations within the smooth fluid paradigm, dark energy has been included in the standard cosmological model, posing a foundational mystery for physics.

In recent years, a number of cosmologists have questioned whether the observations, which have been interpreted as cosmic acceleration, might in fact be accounted for by taking more care in deriving the geometry that comes from averaging the actual inhomogeneous matter distribution. In particular, the geometry which arises from the time evolution of an initial average of the matter distribution does not generally coincide, at a later time, with the average geometry of the full inhomogeneous matter distribution evolved via Einstein's equations [3]. Whether or not the resulting "back-reaction" of inhomogeneities on the average geometry can be large enough to explain effects usually attributed to cosmic acceleration from dark energy in Friedmann–Lemaître–Robertson–Walker (FLRW) models, has been the subject of intense debate. (See [4] and references therein.)

In this *Letter* I will derive the general exact solution to the Buchert equations [3] for the two–scale model introduced in ref. [5], yielding a simple new observationally viable model of the universe. The observational claim is based on my proposal that the debate about dark energy from structure formation [4] can be resolved by careful consideration of the operational interpretation of measurements in cosmology from first principles [5]. This is necessary when averaging an inhomogeneous cosmology, since in general the rods and clocks of observers will be calibrated differently from those at an average location. In writing down average parameters one must define how they are related to our measurements operationally. Assuming the Copernican principle, the fact that we observe an almost isotropic CMB means that other observers should also measure an almost isotropic CMB. However, it does not demand that such observers measure the same mean CMB temperature as we, nor the same angular scale for the Doppler peaks in the anisotropy spectrum. Significant differences can arise due to gradients in spatial curvature and associated gravitational energy.

In general relativity space is dynamical and can carry energy and momentum. By the strong equivalence principle, since the laws of physics must coincide with those of special relativity at a point, it is only internal energy that can be localised in an energy–momentum tensor on the r.h.s. of the Einstein equations. Thus the uniquely relativistic aspects of gravitational energy associated with gradients in spatial curvature, and gradients in the kinetic energy of spatial expansion, cannot be included in the energy momentum tensor, but are at best described by a quasilocal formulation. (For a review, see [6].)

In ref. [5] I propose a quantitative solution to the problem of apparent cosmic acceleration through the technical definition of a *finite infinity* scale, realising a qualitative suggestion of Ellis [7]. Finite infinity replaces the usual notion of spatial infinity in exact asymptotically flat spacetimes, as the fiducial reference point for quasi–local gravitational energy with respect to observers in virialised bound systems. A universal definition of this scale is possible, since the initial expansion rate of the universe was extremely smooth at last scattering, leading to a true critical density, $\rho_{\rm cr}$, as a demarcation between potentially bound and unbound systems. Due to back–reaction, $\rho_{\rm cr}$, does not evolve by the Friedmann equation.

While it has long been understood that averaging an inhomogeneous universe entails the dressing of average cosmological parameters [8] through volume factors which relate to differences in spatial curvature, the proposal of ref. [5] recognises that clock rates can also vary systematically between observers in bound systems within finite infinity, and a volume–average position in freely expanding space, due to differences in gravitational energy. By this means an implicit solution of the Sandage–de Vaucouleurs paradox [5] is possible: the locally defined or bare Hubble parameter, $\bar{H}$, can be uniform even though voids appear to expand faster than the bubble walls which surround them, since cosmic clocks within voids tick faster on account of gravitational energy differences. Since our cosmological observations involve photons exchanged with objects in bound systems, we do not observe clocks in freely expanding space directly. Nonetheless, an ideal comoving observer within a void would measure a somewhat older age of the universe, and an isotropic CMB with a lower mean temperature and an angular anisotropy scale shifted to smaller angles.

Buchert's scheme is somewhat heuristic, since it does not average all of the Einstein equations and requires an extra integrability condition to ensure closure. However, starting from a fully covariant averaging scheme [9], with reasonable cosmological assumptions, the correlation tensor takes the form of a spatial curvature [10], and Buchert's scheme can be realised as a consistent limit [11]. Following ref. [5], the Buchert average constructed here is based on the two scales most relevant to the observed universe: (i) the voids which dominate the universe at the present epoch; and (ii) finite infinity regions containing galaxy clusters within the filaments and bubble walls that surround voids. The local average geometry at the boundary of a finite infinity region is assumed to be spatially flat, with the metric

$$\mathrm{d}s^2_{\mathcal{F}_I} = -\mathrm{d}\tau^2 + a_{\rm w}{}^2(\tau)\left[\mathrm{d}\eta_{\rm w}^2 + \eta_{\rm w}^2\mathrm{d}\Omega^2\right] \,. \qquad (1)$$

Within voids the metric is not given by (1) but is negatively curved, with local scale factor $a_{\rm v}$. We average over the entire present epoch horizon volume, $\mathcal{V} = \mathcal{V}_{\rm i}\bar{a}^3$, where $\bar{a}^3 = f_{\rm vi}a_{\rm v}{}^3 + f_{\rm wi}a_{\rm w}{}^3$; $f_{\rm vi}$ and $f_{\rm wi} = 1 - f_{\rm vi}$ being the respective initial void and wall volume fractions at last scattering, to construct the Buchert average geometry

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \bar{a}^2(t)\,\mathrm{d}\bar{\eta}^2 + A(\bar{\eta},t)\,\mathrm{d}\Omega^2. \qquad (2)$$

Here the area function A is defined by a horizon-volume average [5]. The time–parameter t differs from the wall–time τ of (1) by the mean lapse function $\mathrm{d}t = \bar{\gamma}(\tau)\,\mathrm{d}\tau$. The geometry (2) is not locally isometric to the local geometry in either the walls or void centres.

When the geometry (1) is related to the average geometry (2) by conformal matching of radial null geodesics it may be rewritten

$$\mathrm{d}s^2_{\mathcal{F}_I} = -\mathrm{d}\tau^2 + \frac{\bar{a}^2}{\bar{\gamma}^2}\left[\mathrm{d}\bar{\eta}^2 + r_{\rm w}^2(\bar{\eta},\tau)\,\mathrm{d}\Omega^2\right] \qquad (3)$$

where $r_{\rm w} \equiv \bar{\gamma}\,(1-f_{\rm v})^{1/3}\,f_{\rm wi}{}^{-1/3}\eta_{\rm w}(\bar{\eta},\tau)$. Two sets of cosmological parameters are relevant: those relative to an ideal observer at the volume–average position in freely expanding space using the metric (2), and conventional dressed parameters using the metric (3). The conventional metric (3) arises in our attempt to fit a single global metric (1) to the universe with the assumption that average spatial curvature and local clock rates everywhere are identical to our own, which is no longer true.

The volume–average matter, curvature and kinematic back-reaction parameters are given by $\bar{\Omega}_M = 8\pi G\bar{\rho}_{M0}\bar{a}_0^3/[3\bar{H}^2\bar{a}^3]$, $\bar{\Omega}_k = -k_{\rm v}f_{\rm vi}{}^{2/3}f_{\rm v}{}^{1/3}/[\bar{a}^2\bar{H}^2]$, and $\bar{\Omega}_{\mathcal{Q}} = -\dot{f}_{\rm v}{}^2/[9f_{\rm v}(1-f_{\rm v})\bar{H}^2]$ respectively, where the average curvature is due to the voids only, which are assumed to have $k_{\rm v} < 0$, an overdot denotes a derivative w.r.t. volume–average time, t, and $\bar{H} \equiv \dot{\bar{a}}/\bar{a}$ is the volume–average or bare Hubble parameter. It satisfies

$$\bar{H} = f_{\rm v}H_{\rm v} + f_{\rm w}H_{\rm w}, \qquad (4)$$

where $H_{\rm v} \equiv \dot{a}_{\rm v}/a_{\rm v}$ and $H_{\rm w} \equiv \dot{a}_{\rm w}/a_{\rm w}$ are the regional average expansion rates of voids and walls, *as measured by volume–average clocks*. We define $h_r(t) \equiv H_{\rm w}/H_{\rm v} < 1$. The independent Buchert equations [3], including the integrability condition that ensures their closure, are

$$\bar{\Omega}_M + \bar{\Omega}_k + \bar{\Omega}_{\mathcal{Q}} = 1, \qquad (5)$$

$$\bar{a}^{-6}\partial_t\left(\bar{\Omega}_{\mathcal{Q}}\bar{H}^2\bar{a}^6\right) + \bar{a}^{-2}\partial_t\left(\bar{\Omega}_k\bar{H}^2\bar{a}^2\right) = 0\,. \qquad (6)$$

Conventional dressed parameters defined with respect to the geometry (3), relevant to "wall observers" such as ourselves, do not satisfy a simple relation analogous to (5). The conventional matter density parameter, Ω_M, is expected to take numerical values similar to those we infer in FLRW models. It differs from the bare volume–average density parameter, $\bar{\Omega}_M$, according to $\Omega_M = \bar{\gamma}^3\bar{\Omega}_M$. The mean lapse function is given by

$$\bar{\gamma} = 1 + h_r^{-1}(1-h_r)f_v\,, \qquad (7)$$

as a result of the requirement that the bare, or "locally" measured, expansion rate is uniform. The dressed Hubble parameter that we measure as wall observers, the global average over both walls and voids, is not $\bar{H}$, but

$$H = \bar{\gamma}\bar{H} - \tfrac{\mathrm{d}}{\mathrm{d}t}\,\bar{\gamma} = \bar{\gamma}\bar{H} - \bar{\gamma}^{-1}\,\tfrac{\mathrm{d}}{\mathrm{d}\tau}\,\bar{\gamma}\,. \qquad (8)$$

The Buchert equations (5), (6) may be reduced to the pair of first order equations

$$(1-f_{\rm v})\,\frac{\dot{\bar{a}}}{\bar{a}} - \tfrac{1}{3}\dot{f}_{\rm v} = \sqrt{\bar{\Omega}_{M0}\bar{H}_0^2(1-\epsilon_i)\,(1-f_{\rm v})\,\frac{\bar{a}_0^3}{\bar{a}^3}}\,, \qquad (9)$$

$$\frac{\dot{\bar{a}}}{\bar{a}} + \frac{\dot{f}_{\rm v}}{3f_{\rm v}} = \frac{\bar{H}_0\bar{a}_0}{f_{\rm v}{}^{1/3}\bar{a}}\sqrt{\frac{\bar{\Omega}_{k0}}{f_{\rm v0}^{1/3}} + \bar{\Omega}_{M0}\epsilon_i\frac{\bar{a}_0}{f_{\rm v}{}^{1/3}\bar{a}}}\,, \qquad (10)$$

where $\epsilon_i \ll 1$ is an integration constant, obtained from a first integral [5] of eqs. (5) and (6), namely

$$(1-\epsilon_i)\,\bar\gamma^2\bar\Omega_M(1-f_{\rm v})^{-1}=1.$$

Eqs. (9) and (10) are readily integrated. Firstly we multiply (9) by $\bar H_0^{-1}\,(1-f_{\rm v})^{-2/3}\,\bar a$ to obtain

$$\frac{1}{\bar H_0}\frac{{\rm d}}{{\rm d}t}\left[(1-f_{\rm v})^{1/3}\,\bar a\right]=\sqrt{\frac{\bar\Omega_{M0}(1-\epsilon_i)\bar a_0^3}{(1-f_{\rm v})^{1/3}\,\bar a}}$$

with the solution

$$(1-f_{\rm v})^{1/3}\,\bar a=\bar a_0\left[(1-\epsilon_i)\,\bar\Omega_{M0}\right]^{1/3}\left(\frac32\bar H_0 t\right)^{2/3},\quad(11)$$

where a constant of integration corresponding to the origin of time has been set to zero without loss of generality. Since $f_{\rm wi}{}^{1/3}a_{\rm w}=(1-f_{\rm v})^{1/3}\,\bar a$, we see that $a_{\rm w}=a_{\rm w0}t^{2/3}$, where $a_{\rm w0}\equiv\bar a_0\left[\frac94 f_{\rm wi}{}^{-1}(1-\epsilon_i)\,\bar\Omega_{M0}\bar H_0{}^2\right]^{1/3}$. Thus the local expansion rate within the wall regions is exactly that of an Einstein–de Sitter universe, *but in volume average time not locally measured wall time.*

Finally, we multiply (10) by $\bar a_0^{-1}\bar H_0^{-1}f_{\rm v}{}^{1/3}\bar a$ to obtain

$$\frac{1}{\bar H_0}\frac{{\rm d}u}{{\rm d}t}=\sqrt{\frac{\bar\Omega_{k0}}{f_{\rm v0}^{1/3}}\left(1+\frac{C_\epsilon}{u}\right)},\qquad(12)$$

where $u\equiv f_{\rm v}{}^{1/3}\bar a/\bar a_0=f_{\rm vi}{}^{1/3}a_{\rm v}/\bar a_0$, is proportional to $a_{\rm v}$, and $C_\epsilon\equiv\epsilon_i\bar\Omega_{M0}f_{\rm v0}^{1/3}/\bar\Omega_{k0}$ is a constant which can either be positive, zero or negative depending on the initial value ϵ_i. Integrating (12) we find

$$\sqrt{u(u+C_\epsilon)}-C_\epsilon\ln\left(\left|\frac{u}{C_\epsilon}\right|^{\frac12}+\left|1+\frac{u}{C_\epsilon}\right|^{\frac12}\right)=\frac{\alpha}{\bar a_0}\,(t+t_\epsilon)\qquad(13)$$

where $\alpha=\bar a_0\bar H_0\bar\Omega_{k0}^{1/2}/f_{\rm v0}^{1/6}$, and t_ϵ is a constant which cannot be chosen to be zero without loss of generality, as the time origin was already fixed in determining (11).

Eqs. (11) and (13) constitute the general exact solution to the two–scale Buchert equations (5), (6), regardless of the physical interpretation of observable quantities. In particular, the Buchert equations have been studied by a number of authors without mention as to what the physical interpretation of the time parameter, t, is. Here we will pursue the interpretational framework of ref. [5].

A number of quantities of interest may be found directly. Since $H_{\rm w}=\dot a_{\rm w}/a_{\rm w}=2/(3t)$, and

$$H_{\rm v}=\frac{\dot a_{\rm v}}{a_{\rm v}}=\frac{2}{3t}\sqrt{\frac{(1-f_{\rm v})\,\epsilon_i}{f_{\rm v}(1-\epsilon_i)}\left(1+\frac{u}{C_\epsilon}\right)}$$

it follows that

$$h_r=\frac{H_{\rm w}}{H_{\rm v}}=\sqrt{\frac{(1-\epsilon_i)\bar\Omega_{M0}f_{\rm v0}^{1/3}f_{\rm v}}{\left(\bar\Omega_{k0}u+\bar\Omega_{M0}f_{\rm v0}^{1/3}\epsilon_i\right)(1-f_{\rm v})}}.\qquad(14)$$

Evaluating $[(9)+f_{\rm v}(10)]$ at the present epoch, we obtain the following constraint on parameters

$$\sqrt{(1-\epsilon_i)\bar\Omega_{M0}(1-f_{\rm v0})}+\sqrt{(\bar\Omega_{k0}+\bar\Omega_{M0}\epsilon_i)f_{\rm v0}}=1\,.\qquad(15)$$

This reduces the number of free parameters by one; we may take $\bar\Omega_{k0}$ as dependent, for example. The value of t_ϵ is also determined in terms of the other parameters by evaluating (13) at the present epoch, to give

$$\frac{\bar\Omega_{k0}^{3/2}}{f_{\rm v0}^{1/2}}\bar H_0(t_0+t_\epsilon)=\sqrt{\bar\Omega_{k0}(\bar\Omega_{k0}+\bar\Omega_{M0}\epsilon_i)}$$
$$-\bar\Omega_{M0}\epsilon_i\ln\left[\sqrt{\left|\frac{\bar\Omega_{k0}}{\bar\Omega_{M0}\epsilon_i}\right|}+\sqrt{\left|1+\frac{\bar\Omega_{k0}}{\bar\Omega_{M0}\epsilon_i}\right|}\right],(16)$$

where the age of the universe in volume–average time is

$$t_0=\frac{2}{3\bar H_0}\sqrt{\frac{1-f_{\rm v0}}{(1-\epsilon_i)\bar\Omega_{M0}}}\,,\qquad(17)$$

on account of (11). We observe that the physical interpretation of the void volume–fraction ceases to be physical meaningful in the limit $t\to0$ if $t_\epsilon>0$, as is generally the case. Radiation must be included to describe the universe at early times, and has been omitted here. Here we restrict attention to the matter–dominated epoch.

The general solution is specified by four independent parameters, $\bar H_0$, ϵ_i, $\bar\Omega_{M0}$ and $f_{\rm v0}$. However, two of these may be further eliminated by taking priors [12] at the surface of last scattering consistent with the CMB.

Since eq. (13) is a transcedental equation, the combination of eqs. (11) and (13) only define $\bar a(t)$ and $f_{\rm v}(t)$ implicitly in the general case. Nonetheless, at late times when both t and u are large, all general solutions to (13) tend to the particular solution with $\epsilon_i=0$ and $t_\epsilon=0$. This particular solution is in fact a late–time tracker solution, an attractor which is insensitive to h_{ri} and $f_{\rm vi}$, as long as the observable universe is void–dominated.

As $C_\epsilon=0$ when $\epsilon_i=0$, eq. (13) yields a particularly simple form for the late–time tracker solution. Since $f_{\rm vi}{}^{1/3}a_{\rm v}=f_{\rm v}{}^{1/3}\bar a=\bar a_0 u$, we see that it corresponds to the case in which the void regions expand exactly as a Milne universe in volume–average time, $a_{\rm v}=a_{\rm v0}t$, where $a_{\rm v0}\equiv\bar\Omega_{k0}^{1/2}\bar a_0\bar H_0 f_{\rm v0}^{-1/6}f_{\rm vi}{}^{-1/3}$. It follows that $h_r=H_{\rm w}/H_{\rm v}=2/3$ is a constant for this special solution. If we combine (15) with (14) evaluated at the present epoch, we see that only one of the parameters $\bar\Omega_{M0}$, $\bar\Omega_{k0}$ and $f_{\rm v0}$ is independent, and in fact, $\bar\Omega_{M0}=4(1-f_{\rm v0})/(2+f_{\rm v0})^2$, $\bar\Omega_{k0}=9f_{\rm v0}/(2+f_{\rm v0})^2$.

Consequently, the volume–average scale factor is

$$\bar a=\frac{\bar a_0\big(3\bar H_0 t\big)^{2/3}}{2+f_{\rm v0}}\left[3f_{\rm v0}\bar H_0 t+(1-f_{\rm v0})(2+f_{\rm v0})\right]^{1/3}\qquad(18)$$

for the late–time tracker, while its void fraction is

$$f_{\rm v}=\frac{3f_{\rm v0}\bar H_0 t}{3f_{\rm v0}\bar H_0 t+(1-f_{\rm v0})(2+f_{\rm v0})}\,,\tag{19}$$

and the wall fraction is easily deduced from $f_{\rm w}=1-f_{\rm v}$. In terms of $f_{\rm v}(t)$, the mean lapse function, bare Hubble parameter and bare matter density have the simple forms $\bar\gamma(t)=\frac{3}{2}t\bar H(t)=1+\frac{1}{2}f_{\rm v}$, and $\bar\Omega_M(t)=4(1-f_{\rm v})/(2+f_{\rm v})^2$.

The tracker–solution dressed Hubble parameter (8) is

$$H=\frac{2}{3t}+\frac{f_{\rm v}(4f_{\rm v}+1)}{6t}\,,\tag{20}$$

Other dressed parameters [5] using the metric (3) relevant to galactic wall obsevers include the redshift, z, and luminosity distance, $d_L=\bar\gamma_0^{-1}\bar a_0(1+z)r_{\rm w}$, where $r_{\rm w}=\bar\gamma\,(1-f_{\rm v})^{1/3}\int_t^{t_0}{\rm d}t'/[\bar\gamma(t')(1-f_{\rm v}(t'))^{1/3}\bar a(t')]$. For the tracker solution these respectively satisfy

$$z+1\;=\;\frac{\bar a_0\bar\gamma}{\bar a\bar\gamma_0}=\frac{(2+f_{\rm v})f_{\rm v}{}^{1/3}}{3f_{\rm v0}^{1/3}\bar H_0 t}\,,\tag{21}$$

$$\frac{\bar H_0 d_L}{(1+z)^2}\;=\;\left(\bar H_0 t\right)^{\frac{2}{3}}\int_t^{t_0}\frac{2\bar H_0{\rm d}t'}{(2+f_{\rm v}(t'))(\bar H_0 t')^{2/3}}\,.\tag{22}$$

This last integral can be given in a simple closed analytic form. These expressions are given in terms of volume–average time. To convert to wall time relevant to galactic observers, we perform the integral, $\tau=\int_0^t{\rm d}t\,\bar\gamma^{-1}$, giving

$$\tau=\tfrac{2}{3}t+\frac{4\Omega_{M0}}{27f_{\rm v0}\bar H_0}\ln\left(1+\frac{9f_{\rm v0}\bar H_0 t}{4\Omega_{M0}}\right)\,,\tag{23}$$

where $\Omega_{M0}=\frac{1}{2}(1-f_{\rm v0})(2+f_{\rm v0})$ is the present epoch dressed matter density in the case of the tracker solution. In (20)–(22) the parameter t should be considered to be implicitly defined by (23) in terms of wall time, τ. At late times as $t\to\infty$, $\tau\sim\frac{2}{3}t$, so that $H\sim\frac{3}{2}t^{-1}\sim\tau^{-1}$, as for an empty Milne universe.

The age of the universe in volume–average time is $t_0=(2+f_{\rm v0})/(3\bar H_0)$, and

$$\tau_0=\frac{2(2+f_{\rm v0})}{9\bar H_0}\left[1+\frac{(1-f_{\rm v0})}{3f_{\rm v0}}\ln\left(\frac{2+f_{\rm v0}}{2(1-f_{\rm v0})}\right)\right]\,,\tag{24}$$

in wall time. The dressed Hubble constant we measure, H_0, is related to the bare Hubble constant, $\bar H_0$, by

$$H_0=\frac{(4f_{\rm v0}^2+f_{\rm v0}+4)\bar H_0}{2(2+f_{\rm v0})}\,.\tag{25}$$

Finally, the volume–average tracker–solution deceleration parameter is $\bar q=\frac{1}{2}\bar\Omega_M+2\bar\Omega_Q=2\,(1-f_{\rm v})^2/(2+f_{\rm v})^2$, which begins close to the Einstein–de Sitter value, $\bar q\sim\frac{1}{2}$, when $f_{\rm v}$ is small, and approaches $\bar q\to0^+$ at late times, but remains positive at all times. Thus a volume–average observer in freely expanding space detects no cosmic acceleration. Nonetheless, a bound system observer measures an effective dressed deceleration parameter

$$q=\frac{-\,(1-f_{\rm v})\,(8f_{\rm v}{}^3+39f_{\rm v}{}^2-12f_{\rm v}-8)}{\left(4+f_{\rm v}+4f_{\rm v}{}^2\right)^2}\,,\tag{26}$$

which also begins at the Einstein–de Sitter value, $q\sim\frac{1}{2}$, for small $f_{\rm v}$ but then changes sign at epoch when $f_{\rm v}\simeq0.5867$, at a zero of the cubic in (26). Apparent acceleration reaches a maximum when $f_{\rm v}\simeq0.7736$ when $q\simeq-0.043$, and then $q\to0^-$ at late times. Thus a wall observer registers a late–time evolution which again is close to that of a Milne universe, but this time with *apparent acceleration.*

While the initial conditions of the CMB require that $h_r\to1$ at last scattering, for observationally relevant initial conditions, the general solution (11), (13) approaches the tracker solution to within 1% by a redshift of $z\sim37$. Thus the tracker solution can be used as a very reliable approximation all the way back to the epoch of reionization. It is effectively the simplest viable generalisation of the Einstein–de Sitter and Friedmann models, which incorporates back–reaction, and is potentially of great utility. Detailed cosmological parameter fits are presented elsewhere [12], and demonstrate that the present *fractal bubble model* [5] is at the very least a serious contender for quantitatively solving the problem of dark energy purely within general relativity. The analytic solution presented here, and its simple tracker limit, should therefore provide the basis for many future cosmological tests.

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[1] F. Hoyle and M.S. Vogeley, Astrophys. J. **566**, 641 (2002); Astrophys. J. **607**, 751 (2004).

[2] A.V. Tikhonov and I.D. Karachentsev, Astrophys. J. **653** (2006) 969; L. Rudnick, S. Brown and L.R. Williams, arXiv:0704.0908.

[3] T. Buchert, Gen. Relativ. Grav. **32**, 105 (2000).

[4] S. Räsänen, JCAP **11**, 003 (2006); M.N. Célérier, astro-ph/0702416; A.A. Coley, arXiv:0704.1734; T. Buchert, AIP Conf. Proc. **910**, 361 (2007); arXiv:0707.2153.

[5] D.L. Wiltshire, New J. Phys. **9**, 377 (2007).

[6] L.B. Szabados, Living Rev. Rel. **7**, 4 (2004).

[7] G.F.R. Ellis, in B. Bertotti, F. de Felice and A. Pascolini (eds), *General Relativity and Gravitation*, (Reidel, Dordrecht, 1984) pp. 215–288.

[8] T. Buchert and M. Carfora, Phys. Rev. Lett. **90**, 031101 (2003); Class. Quantum Grav. **19**, 6109 (2002).

[9] R.M. Zalaletdinov, Gen. Relativ. Grav. **24**, 1015 (1992); **25**, 673 (1993).

[10] A.A. Coley, N. Pelavas and R.M. Zalaletdinov, Phys. Rev. Lett. **95** (2005) 151102; A.A. Coley and N. Pelavas, Phys. Rev. **D 74**, 087301 (2006); **D 75**, 043506 (2007).

[11] A. Paranjape and T.P. Singh, Phys. Rev. **D 76**, 044006 (2007).

[12] B.M. Leith, S.C.C. Ng and D.L. Wiltshire, arxiv:0709.2535.