Physics 581, Quantum Optics II
Problem Set #4
Due: Thurs. March 26, 2020

Problem 1: Nonclassical light generation via the Kerr effect. (15 points)
In the classical (optical) Kerr effect, the index of refraction is proportional to the intensity. The quantum optical description is via the Hamiltonian,

$$\hat{H} = \frac{\hbar\chi^{(3)}}{2} : \hat{I}^2 := \frac{\hbar\chi^{(3)}}{2} \hat{a}^{\dagger 2}\hat{a}^2 .$$

(a) Suppose we inject a strong coherent state into a nonlinear fiber with Kerr response. *Linearize* this Hamiltonian about the mean field via the substitution $\hat{a} = \alpha + \hat{b}$, and keep terms only up to quadratic order in $\hat{b}$ and $\hat{b}^\dagger$. Show that the resulting leads to squeezing.

(b) Now let's go beyond the linear approximation. Start with an initial coherent state $|\alpha\rangle$. Show that for a long time such that $\chi^{(3)} t = \pi$, the state becomes a Schrödinger cat, $\left(e^{i\pi/4}|-i\alpha\rangle + e^{-i\pi/4}|i\alpha\rangle\right)/\sqrt{2}$.

Problem 2: Toward an optical "Schrödinger cat state." (25 points)

Creating a "Schrödinger cat state," e.g. $\left|cat_\phi(\alpha_0)\right\rangle = N\left(\left|\alpha_0\right\rangle + e^{i\phi}\left|-\alpha_0\right\rangle\right)$, where the normalization $N = 1/\sqrt{2\left(1+\cos\phi\, e^{-2|\alpha_0|^2}\right)}$, is a challenging task in the optical regime because we do not have sufficient nonlinearity with low loss (in the microwave regime, cavity and circuit QED has achieved this – more on that in another problem). Producing something close to such a state for applications in Quantum Information Processing has been an important goal.

Consider a squeezed single photon Fock state, $|r,1\rangle \equiv \hat{S}(r)|1\rangle$

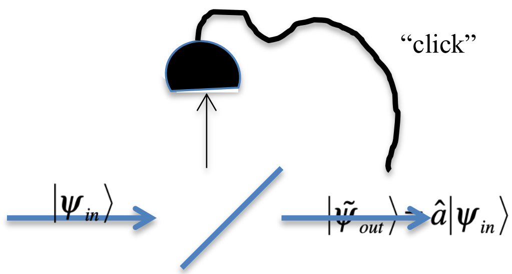
(a) Show that the Wigner function of this state is $W(\alpha) = -\frac{2}{\pi} e^{-2|b|^2} L_1(4|b|^2)$ where $b = \alpha^* \cosh r + \alpha \sinh r$ and L_1 is the first-order Laguerre polynomial. Plot W.

(b) Show that the fidelity between the "odd" cat-state and the squeezed Fock state is

$$F(r,\alpha_0,\pi) \equiv \left|\langle cat_\pi(\alpha_0)|r,1\rangle\right|^2 = \frac{2\alpha_0^2 \exp\left[\alpha_0^2\left(\tanh r - 1\right)\right]}{(\cosh r)^3\left(1-\exp[-2\alpha_0^2]\right)}$$ (where a_0 is real).

(c) Make a surface plot of F as a function of r and α_0. Under what parameters can the squeezed Fock state well approximate the cat state?

While the squeezed Fock state can approach cat state, squeezing a single photon state is not easy to achieve either. The output of a nonlinear optical processes is typically a squeezed vacuum. This is a Gaussian state, which is classical if we perform only homodyne measurements. However, if we have access to other resources, such a photon counting, we can transform this into a non-Gaussian, fully quantum resource. Consider the following experiment:



The light is incident on a highly transmitting beam splitter. Rarely one photon is reflected and detected. Conditioned on that "click," the output state has one of the photons annihilated. This state is "post selected" and the probability of producing it is rare. Nonetheless, this is a highly non-Gaussian operation.

The state produced is a "photon subtracted squeezed state." This operation is non-Unitary, so the post-measurement state is $|\psi_{out}\rangle = \hat{a}\hat{S}(r)|0\rangle / \left\|\hat{a}\hat{S}(r)|0\rangle\right\|$.

(d) Show that $|\psi_{out}\rangle = \hat{S}(r)|1\rangle$, the squeezed Fock state.

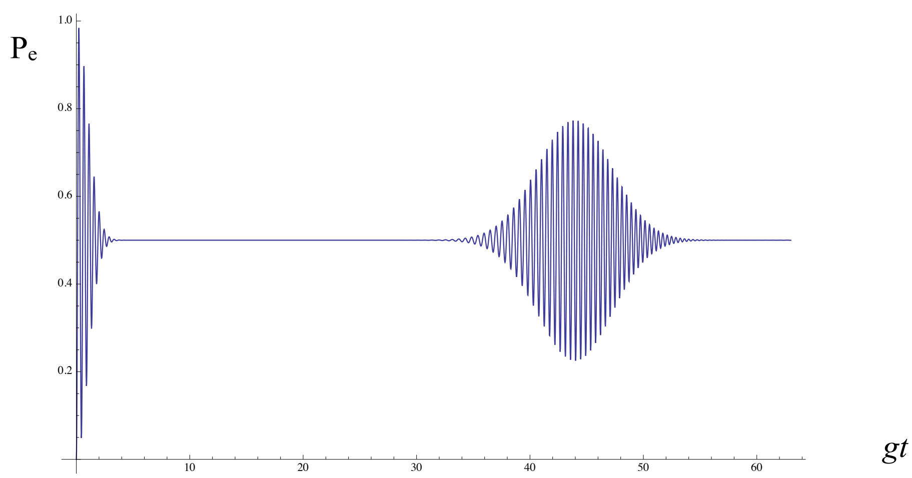
Problem 3: Entanglement and the Jaynes-Cummings Model (30 points)

One the most fundamental paradigms in quantum optics is the coupling of a two-level atom to a single mode of the quantized electromagnetic field. In the rotating wave approximation, this is governed by the Jaynes-Cummings model (JCM),

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_0 \frac{\hat{\sigma}_z}{2} + \hbar g\left(\hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-\right).$$

This is a bipartite system with tensor product Hilbert space for the atom and field, $\mathcal{H}_{AF} = h_A \otimes h_F$, where h_A is the two-dimensional Hilbert space of the two-level atom, and h_F is the infinite dimensional Hilbert space of the harmonic oscillator that describes the mode. The goal of this problem is to understand the entanglement between the atom and mode, generated by the JCM.

Last semester, we studied how this leads to collapse and revival of Rabi oscillation that follows from an initial product state with the field in a coherent state and the atom in, e.g., the ground state $|\Psi(0)\rangle_{AF} = |g\rangle_A \otimes |\alpha\rangle_F$. The probability to find the atom in the excited state oscillates as shown (here for $\langle n\rangle = |\alpha|^2 = 49$)



The collapse is due to the variation of the quantum Rabi oscillations with different number; the revival is uniquely a quantum effect arising from the discreteness of the quantized field, occurring at a time $gt_r \approx 2\pi\sqrt{\langle n\rangle}$ for large $\langle n\rangle$.

(a) Let us consider the case on resonance with $\omega_c = \omega_0$. Show that in the interaction picture, the state at time *t* the joint state takes the form

$$|\Psi(t)\rangle_{AF} = |g\rangle_A \otimes |C(t)\rangle_F + |e\rangle_A \otimes |S(t)\rangle_F$$

where $|C(t)\rangle_F = \sum_{n=0}^{\infty} c_n \cos(\sqrt{n}gt)|n\rangle$, $|S(t)\rangle_F = -i\sum_{n=0}^{\infty} c_{n+1} \sin(\sqrt{n+1}gt)|n\rangle$,

$$c_n = (\alpha^n / \sqrt{n!}) e^{-|\alpha|^2/2} \ .$$

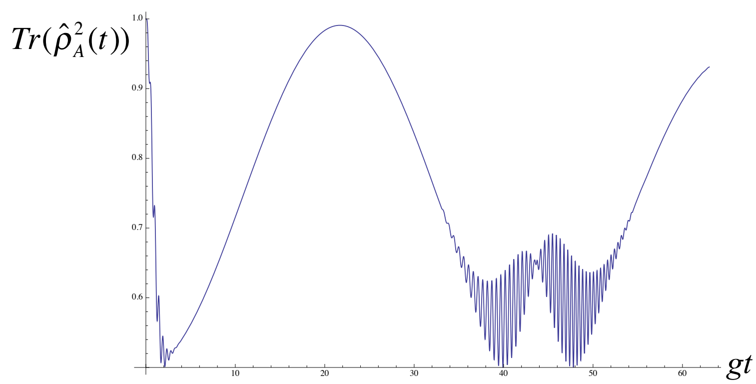
Note $|C(t)\rangle_F$, $|S(t)\rangle_F$ are not normalized, nor are they orthogonal.

(b) Show that the marginal state of the atom in the $\{|g\rangle, |e\rangle\}$ basis is

$$\hat{\rho}_A(t) = \begin{bmatrix} \langle C(t)|C(t)\rangle & \langle C(t)|S(t)\rangle \\ \langle S(t)|C(t)\rangle & \langle S(t)|S(t)\rangle \end{bmatrix} = \frac{1}{2}\left(\hat{1} + \vec{Q}(t)\cdot\hat{\vec{\sigma}}\right).$$

Write an expression for Bloch vector $\vec{Q}(t)$.

(c) Write the purity of the marginal (a measure of the entanglement between the atom and field), in terms of the Bloch vector. Numerically calculate this and plot as a function of time for $\langle n\rangle = |\alpha|^2 = 49$. Your graph should look like



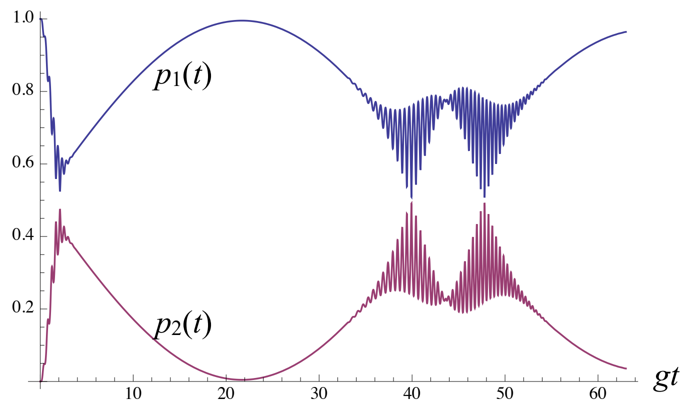
This plots shows a few surprising features. During the collapse the atom and field become highly entangled, as indicated by the rapid degree in the atomic purity. However, at half the revival time, $gt_r/2 \approx \pi\sqrt{\langle n\rangle}$, when the inversion looks to be flat, the purity returns to near unity, indicating that the atom and field become *separable*. The atom and field then become re-entangled. When the Rabi oscillations once again revive, the purity again increases, but nowhere near to unity. Our goal now is to use the Schmidt decomposition to understand this.

(d) Given the initial pure state of the joint system and the unitary evolution according to the JCM, we know that at all times we can express the state in terms of Schmidt decomposition.

$$|\Psi(t)\rangle_{AF} = \sum_{\mu=1}^{2} \sqrt{p_\mu(t)}\,|u_\mu(t)\rangle_A \otimes |v_\mu(t)\rangle_F \ .$$

Note, even though the field mode is infinite dimensional, the maximum Schmidt number is 2.

Express the two values of $p_\mu(t)$ in terms of the Bloch vector $\vec{Q}(t)$. Calculate numerically at plot as function of time. Your graphs should look like the following:



Comment on this and what it means for the entanglement.

(e) We can find the Schmidt vectors by the following procedure.

- Find the atomic Schmidt vectors $\left\{ |u_\mu(t)\rangle_A \right\}$ as the eigenvectors of the marginal state $\hat{\rho}_A(t)$ in the standard basis $\{|g\rangle, |e\rangle\}$.
- Using $|\Psi(t)\rangle_{AF} = |g\rangle_A \otimes |C(t)\rangle_F + |e\rangle_A \otimes |S(t)\rangle_F = \sum_{\mu=1}^{2} \sqrt{p_\mu(t)}\, |u_\mu(t)\rangle_A \otimes |v_\mu(t)\rangle_F$, find an expres-sion for the two Schmidt vectors of the field $\left\{ |v_\mu(t)\rangle_F \right\}$ in terms of $|C(t)\rangle_F, |S(t)\rangle_F$, $p_\mu(t)$.

(f) We can see the (approximate) separation between atom and field at half the revival time for large $\langle n \rangle$ as follows. Show that in this limit,

$$g\sqrt{n+1}\,t_r/2 \approx g\sqrt{n}\,t_r/2 + \pi/2, \quad c_{n+1} \approx e^{-i\phi} c_n, \text{ where } c_n = (\alpha^n/\sqrt{n!})e^{-|\alpha|^2/2} \text{ and } \alpha = \sqrt{\langle n \rangle} e^{i\phi}.$$

Using this, show that

$$|\Psi(t_r/2)\rangle_{AF} \approx \left(|g\rangle_A - ie^{-i\phi}|e\rangle_A \right) \otimes |C(t_r/2)\rangle_F .$$

Thus we see that the system is separable, with the atom in an equal superposition depending on the phase of the coherence state.

(g) Extra credit (5 points): More generally show that if $\left|\Psi(0)\right\rangle_{AF} = \left(a\left|g\right\rangle_A + b\left|e\right\rangle_A\right)\otimes\left|\alpha\right\rangle_F$

$$\left|\Psi(t_r/2)\right\rangle_{AF} \approx \left(\left|g\right\rangle_A - ie^{-i\phi}\left|e\right\rangle_A\right)\otimes\left(a\left|C(t_r/2)\right\rangle_F + b\left|S(t_r/2)\right\rangle_F\right)$$

This result shows that *regardless of the atomic initial state*, at half the revival time, the atom goes to the same state. The information about the initial atomic superposition is transferred to the field in a kind of "swap gate." For large a, the two field states are macroscopically distinguishable. This is kind of "Schrödinger cat".