Physics 581.008 : Quantum Optics II
Lecture 2: Squeezed States (Fundamentals)

So far we have encountered two phenomena that are "smoking guns" for nonclassical light: sub Poissonian photon statistics and photon anti-bunching. Both of these phenomena are incompatible with a model of light as a statistical mixture of coherent states, which arises from a classical current source. These are associated with discrete variables of the field – the photons – and are exhibited in photon counting experiments. In this lecture we consider the first example of a nonclassical phenomenon associated with the continuous variables of the field – squeezing of the quadrature uncertainty of the waves' complex amplitude.

Such phenomena are exhibited in phase-dependent detection of the field, as thus not associated with direct photon counting as we know from the number-phase uncertainty relation

Continuous Variable Algebra Review

Complex amplitude: Phasor $\alpha = \dfrac{X + iP}{\sqrt{2}} = A e^{i\phi}$

$$X = \sqrt{2}\,\mathrm{Re}(\alpha) = \frac{\alpha + \alpha^*}{\sqrt{2}}, \qquad P = \sqrt{2}\,\mathrm{Im}(\alpha) = \frac{\alpha - \alpha^*}{i\sqrt{2}}$$

SHO: $$\alpha(t) = \alpha(0)\, e^{-i\omega t} = \underbrace{\tfrac{1}{\sqrt{2}}\left[X(0)\cos\omega t + P(0)\sin\omega t\right]}_{\frac{1}{\sqrt{2}}\left(X(t)\right.} + \underbrace{\tfrac{i}{\sqrt{2}}\left[P(0)\cos\omega t - X(0)\sin\omega t\right]}_{\left. + \; i \; P(t)\right)}$$

Quantized: $\alpha \to \hat{a}$ $\qquad X \to \hat{X}$ $\qquad P \to \hat{P}$ $\qquad |\alpha|^2 \to \hat{n}$

$$[\hat{X}, \hat{P}] = i, \qquad [\hat{a}, \hat{a}^\dagger] = 1, \qquad [\hat{n}, \hat{a}] = -\hat{a}$$

Phase space Rotation Operator: $\hat{R}(\theta) = e^{-i\theta \hat{a}^\dagger \hat{a}} = e^{-i\theta \hat{n}}$

$$\hat{R}^\dagger(\theta)\, \hat{a}\, \hat{R}(\theta) = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \underbrace{[\hat{n}, [\hat{n}, \cdots [\hat{n}, \hat{a}]]]^{(n)}}_{(-1)^n \hat{a}} = e^{-i\theta}\hat{a}$$

$$\hat{R}^\dagger(\theta) \hat{X} \hat{R}(\theta) = \frac{e^{-i\theta}\hat{a} + e^{+i\theta}\hat{a}^\dagger}{\sqrt{2}} = \hat{X}\cos\theta + \hat{P}\sin\theta \equiv \hat{X}_\theta$$

$$\hat{R}^\dagger(\theta) \hat{P} \hat{R}(\theta) = \frac{e^{-i\theta}\hat{a} - e^{+i\theta}\hat{a}^\dagger}{i\sqrt{2}} = -\hat{X}\sin\theta + \hat{P}\cos\theta \equiv \hat{P}_\theta$$

Note: $\hat{P}_\theta = \hat{X}_{\theta+\frac{\pi}{2}}$, $[\hat{X}_\theta, \hat{P}_\theta] = i$ (In the literature, we sometimes refer to $\hat{X}_1 \equiv \hat{X}$, $\hat{X}_2 \equiv \hat{P}$, $[\hat{X}_1, \hat{X}_2] = i$)

Phase space Displacement Operator $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-i(x\hat{P} - p\hat{X})}$

$$\hat{D}^\dagger(\alpha)\, \hat{a}\, \hat{D}(\alpha) = \sum_n \frac{1}{n!} [\alpha^*\hat{a} - \alpha\hat{a}^\dagger, [\cdots, \hat{a}]]^{(n)} = \hat{a} + \alpha$$

$$\hat{D}^\dagger(\alpha) \hat{X} \hat{D}(\alpha) = \hat{X} + \sqrt{2}\,\mathrm{Re}(\alpha), \quad \hat{D}^\dagger(\alpha) \hat{P} \hat{D}(\alpha) = \hat{P} + \sqrt{2}\,\mathrm{Im}(\alpha)$$

Coherent state

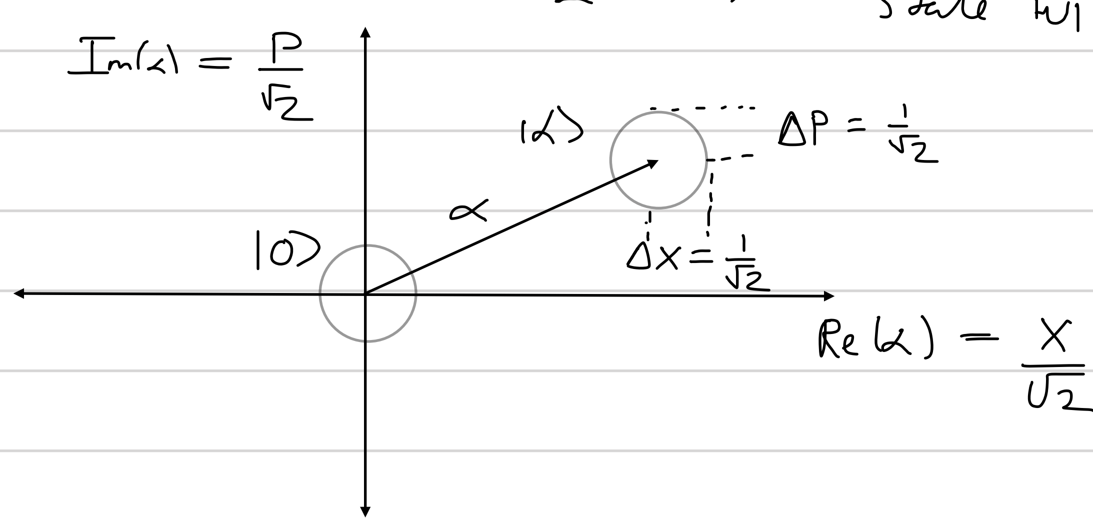
$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle, \quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

vacuum

$$\langle\alpha|\hat{X}|\alpha\rangle = \sqrt{2}\,\mathrm{Im}(\alpha), \quad \langle\alpha|\hat{P}|\alpha\rangle = \sqrt{2}\,\mathrm{Im}(\alpha)$$

$$\langle\alpha|\Delta\hat{X}^2|\alpha\rangle = \langle\alpha|\Delta\hat{P}^2|\alpha\rangle = \frac{1}{2}$$

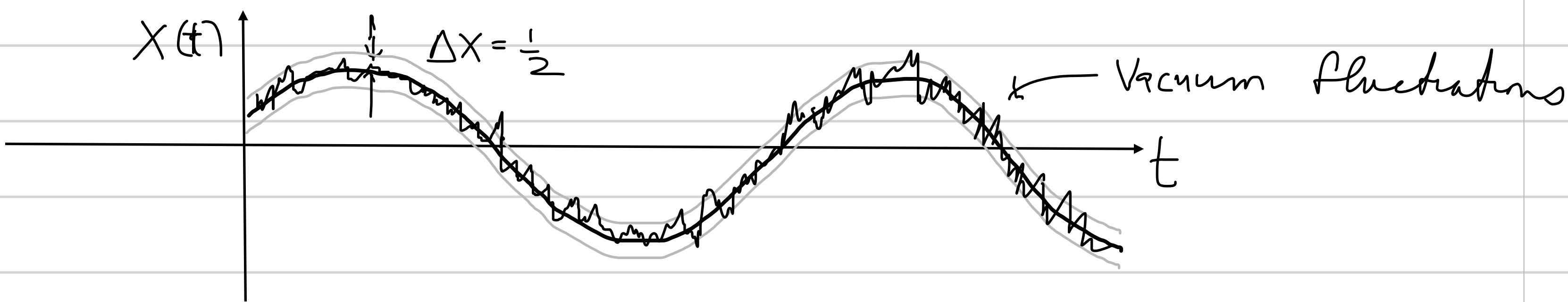
$|0\rangle$ is a coherent state with $\alpha = 0$



If $|\psi(0)\rangle = |\alpha\rangle$, for free SHO, $|\psi(t)\rangle = e^{-i\omega t \hat{a}^\dagger \hat{a}} |\psi(0)\rangle$

$$\Rightarrow |\psi(t)\rangle = e^{-i\omega t \hat{n}} \hat{D}(\alpha) |0\rangle = \hat{D}(\alpha e^{-i\omega t}) |0\rangle = |\alpha e^{-i\omega t}\rangle$$

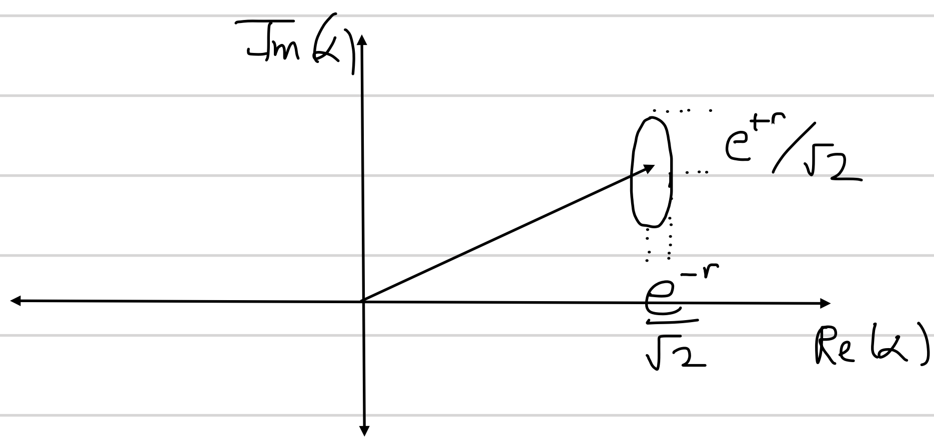
Direct-space picture



Squeezed State Definition

Minimum uncertainty state of a harmonic oscillator, with unequal variances in conjugate quadratures:

e.g. $\Delta X = \frac{e^{-r}}{\sqrt{2}}$, $\Delta P = \frac{e^{+r}}{\sqrt{2}}$



For such a state, the variance in the X-quadrature is "squeezed" below the vacuum level. A squeezed state is nonclassical in the sense that it cannot be produced by a classical light source — it cannot be expressed as a statistical mixture of coherent states:

Proof: $\Delta X^2 = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 = \frac{1}{2}\left(\langle (\hat{a}+\hat{a}^\dagger)^2 \rangle - \langle \hat{a}+\hat{a}^\dagger \rangle^2 \right)$

$$(\hat{a}+\hat{a}^\dagger)^2 = \hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger = \,:(\hat{a}+\hat{a}^\dagger)^2:\, + 1$$

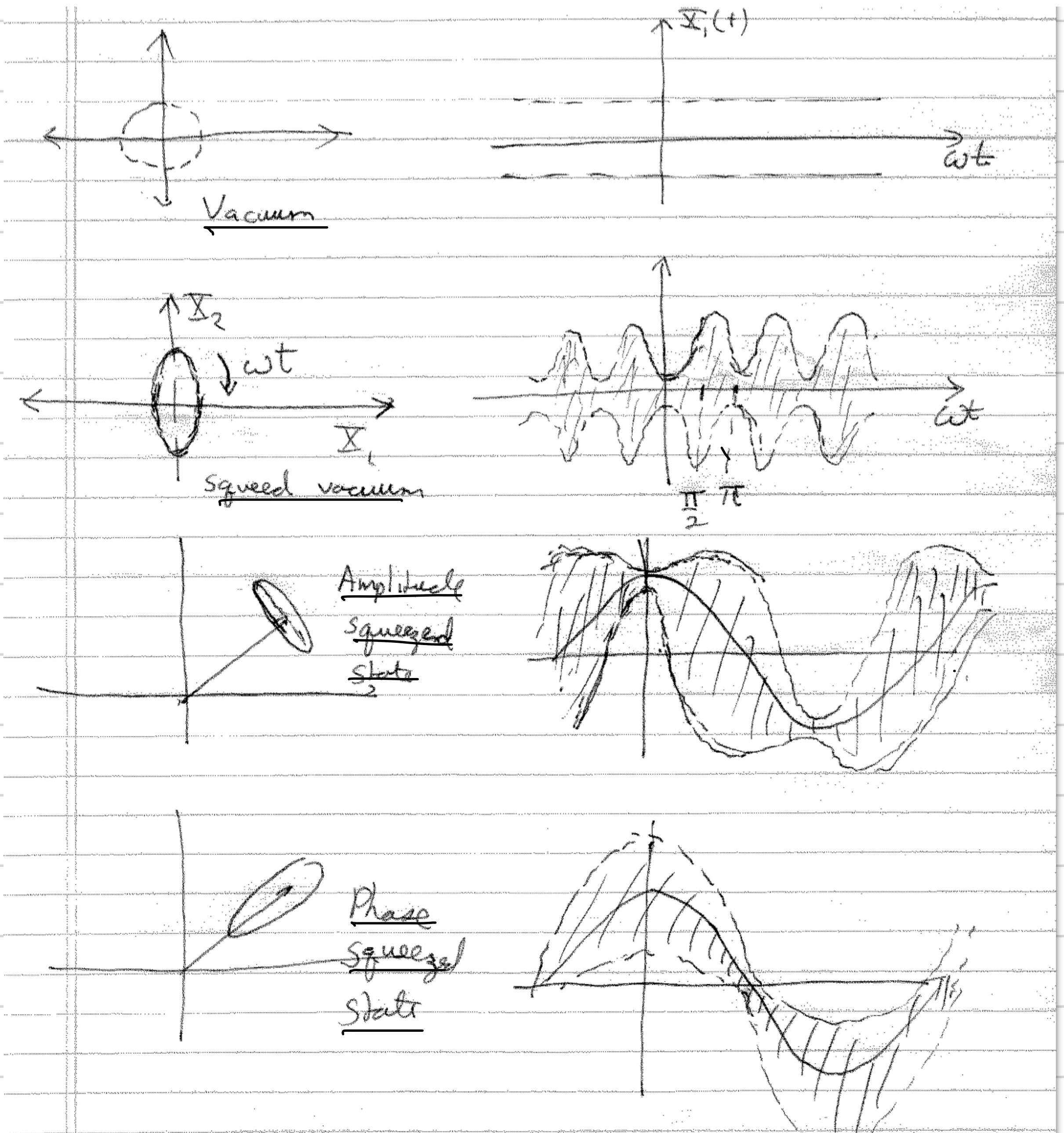
$$\Rightarrow \langle (\hat{a}+\hat{a}^\dagger)^2 \rangle = \int d^2\alpha \, P(\alpha) \, (\alpha+\alpha^*)^2 = \overline{(\alpha+\alpha^*)^2}$$ ← average over statistical ensemble

$$\langle \hat{a}+\hat{a}^\dagger \rangle^2 = \left(\int d^2\alpha \, P(\alpha) \, (\alpha+\alpha^*) \right)^2 = \left(\overline{\alpha+\alpha^*} \right)^2$$

$$\Rightarrow \Delta X^2 = 2\, \overline{\Delta \mathrm{Re}(\alpha)^2} + \frac{1}{2} \geq \frac{1}{2}$$ ← Vacuum level q.e.d.

A classically, noisy current source can only increase the noise level above the vacuum level.

Different varieties of squeezed states



See C M Caves, PRD 23, 1693 (1981).

Squeezed State have application in reducing the fundamental effects of quantum noise in communication and measurement protocols. Historically, squeezed state were introduced by C.M. Caves as a mechanism to increase the signal-to-noise of optical interferometers being designed to detect the tiny forces of gravity waves. This is being implemented in the current generation of LIGO (Laser Interferometer Gravitational-Wave Observatory).

The Squeezing Operator

Consider the unitary operator $\hat{S}(r) \equiv \exp\left\{\frac{1}{2}(r\hat{a}^2 - r\hat{a}^{\dagger 2})\right\}$

$$\hat{S}^\dagger(r)\,\hat{a}\,\hat{S}(r) = e^{\hat{A}}\,\hat{a}\,e^{-\hat{A}} = \hat{a} + [\hat{A},\hat{a}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{a}]] + \cdots,$$

$$\hat{A} = -\frac{r}{2}\hat{a}^2 + \frac{r^2}{2}\hat{a}^{\dagger 2}$$

$$[\hat{A},\hat{a}] = -r\hat{a}^\dagger, \qquad [\hat{A},[\hat{A},\hat{a}]] = r^2\hat{a}, \qquad \text{etc.}$$

$$\Rightarrow \hat{S}^\dagger(r)\,\hat{a}\,\hat{S}(r) = \left(\sum_{n\ \text{even}} \frac{r^n}{n!}\right)\hat{a} - \left(\sum_{n\ \text{odd}} \frac{r^n}{n!}\right)\hat{a}^\dagger$$

$$\boxed{\begin{aligned} \hat{S}^\dagger(r)\,\hat{a}\,\hat{S}(r) &= \cosh r\,\hat{a} - \sinh r\,\hat{a}^\dagger \\ \hat{S}^\dagger(r)\,\hat{a}^\dagger\hat{S}(r) &= \cosh r\,\hat{a}^\dagger - \sinh r\,\hat{a} \end{aligned}}$$

This unitary transformation is known as a <u>Bogoliubov transformation</u>, first studied in the context of elementary excitations relative to a Bose-Einstein condensate.

<u>Note</u>: The Bogoliubov transformation is like a rotation between $\hat{a}$ and $\hat{a}^\dagger$ by an imaginary angle - intimate relationship with the Lorentz Group.

Transformation on the quadratures

$$\hat S^\dagger \hat X \hat S = \frac{1}{\sqrt2}\left(\hat S^\dagger \hat a \hat S + \hat S^\dagger \hat a^\dagger \hat S\right) = \frac{1}{\sqrt2}\left(c\hat a - s\hat a^\dagger + c\hat a^\dagger - s\hat a\right)$$

$c = \cosh r$, $s \equiv \sinh r$

$$\Rightarrow \boxed{\hat S^\dagger \hat X \hat S = (c-s)\left(\frac{\hat a+\hat a^\dagger}{\sqrt2}\right) = e^{-r}\hat X}$$

$$\boxed{\text{Similarly} \quad \hat S^\dagger \hat P \hat S = e^{+r}\hat P}$$

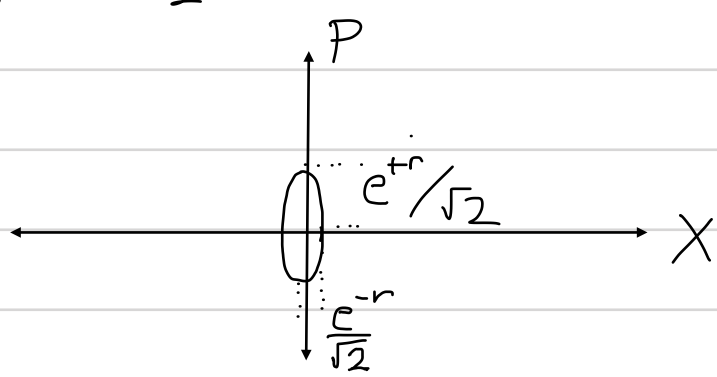
"Squeezed Vacuum"

$$|0_r\rangle = \hat S(r)|0\rangle \quad \Rightarrow \quad \langle 0_r|\hat a|0_r\rangle = 0 \quad (\text{mean-field} = 0)$$

$$\langle 0_r|\Delta\hat X^2|0_r\rangle = \langle 0_r|\hat X^2|0_r\rangle - \cancel{\langle 0_r|X|0_r\rangle^2}^{\,0} = \langle 0|(\hat S^\dagger\hat X\hat S)^2|0\rangle$$

$$\Rightarrow \langle\Delta\hat X^2\rangle_r = e^{-2r}\langle 0|\hat X^2|0\rangle = \frac12 e^{-2r}$$

Similarly $\langle\Delta\hat P^2\rangle_r = \frac12 e^{+2r}$



Note: $\langle\hat n\rangle_r = \langle 0_r|\hat a^\dagger\hat a|0_r\rangle = \langle 0|\hat a_r^\dagger \hat a_r|0\rangle$

$$= \langle 0|(c\hat a - s\hat a^\dagger)(c\hat a^\dagger - s\hat a)|0\rangle = \sinh^2 r \neq 0$$

$\Rightarrow$ Squeezed Vacuum has photons

The reduction in quadrature fluctuation due to coherence between vacuum and photon pairs.

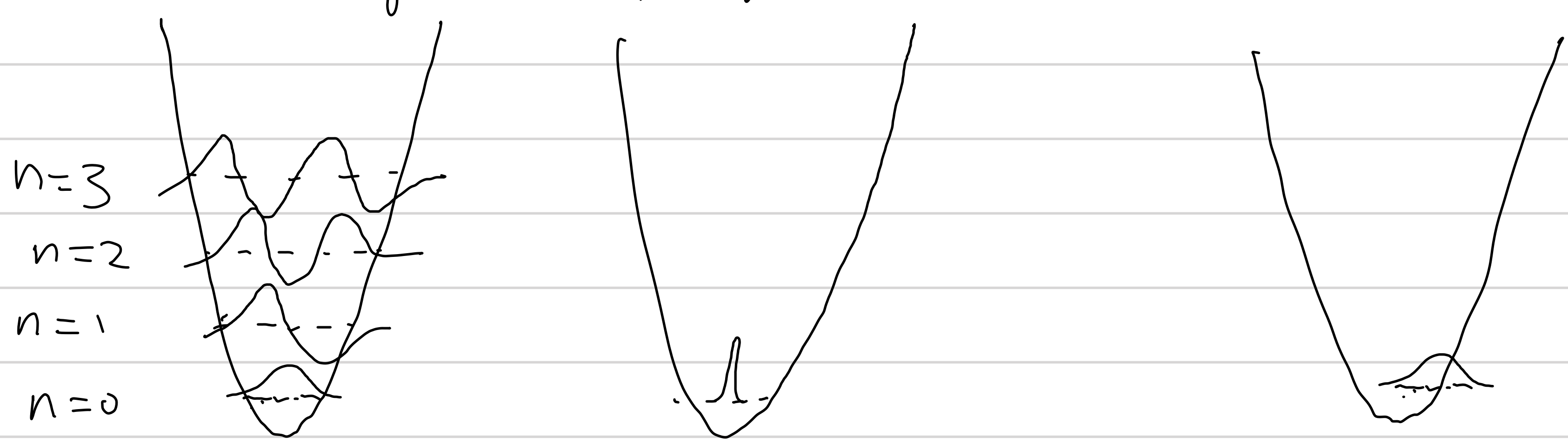
$$|0_r\rangle \approx |0\rangle - \frac r2 \hat a^{\dagger 2}|0\rangle = |0\rangle - \frac{r}{\sqrt2}|2\rangle \quad (r\ \text{small}),$$

$$\langle 0_r|\hat X^2|0_r\rangle = \frac12\langle 0_r|\hat a^2+\hat a^{\dagger 2}+\hat a^\dagger\hat a+\hat a\hat a^\dagger|0_r\rangle \approx \frac{1-2r}{2} \quad (r\ \text{small})$$

We can gain further insight into nature of squeezing from a "wave mechanics" perspective. Consider the "wave functions" of the Fock states

$$u_n(X) = \langle X | n \rangle = N_n \underbrace{H_n}_{\text{Hermite Polynomial}}(X)\, e^{-X^2/2}$$

Parity $(-1)^n$, operator $\hat{\Pi} = (-1)^{\hat{n}}$



Squeezed vacuum:

$$u_r(x) = \langle x | S(r) | 0 \rangle :$$

$$= \langle x | e^{\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})} | 0 \rangle = \sum_{n\,\text{even}} c_n u_n(x)$$

Coherent state:

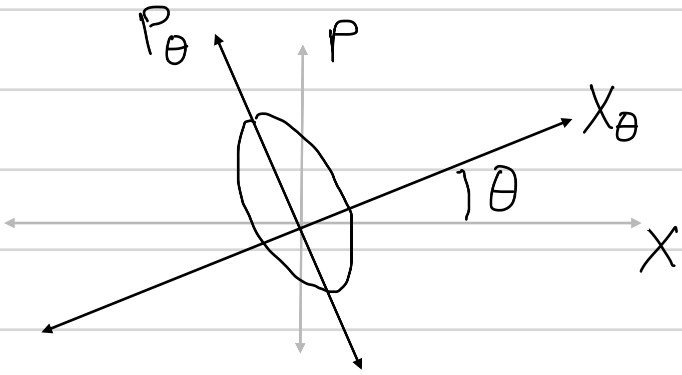
$$u_\alpha(x) = \langle x | \hat{D}(\alpha) | 0 \rangle$$

$$= \sum_n e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} u_n(x)$$

the squeezed vacuum state is a positive parity state, containing only even photon numbers (n even) $\Rightarrow$ Squeezing arises from photon pair correlations as we will see in more detail. In contrast, a coherent state involve essential uncorrelated photons; this is the most classical of pure states.

Aside: Note $\langle X | n \rangle$ is not the wave function of the photon in in position space, because $\hat{X}$ is not the position-of-the photon operator. It is the quadrature operator. $\langle X | n \rangle$ is the probability amplitude for a quadrature measurement of X.

More General Case:



Squeezed vacuum with different quadratures.

$\hat R(-\theta)\,\hat S(r)\,|0\rangle$

$\hat R^\dagger(\theta)\,\hat S(r)\,\hat R(\theta)\,|0\rangle$

$$\hat R^\dagger(\theta)\,\hat S(r)\,\hat R(\theta) = \exp\left\{\frac{r}{2}\left[(\hat R^\dagger \hat a\,\hat R)^2 - (\hat R^\dagger \hat a\,\hat R)^2\right]\right\}$$

$$= \exp\left\{\frac{1}{2}\left[re^{-i2\theta}\,\hat a^2 - re^{i2\theta}\,\hat a^{\dagger 2}\right]\right\} \equiv \hat S(\zeta)$$

$$\boxed{\hat S(\zeta) \equiv \exp\left\{\frac{1}{2}\left(\zeta^*\hat a^2 - \zeta\,\hat a^{\dagger 2}\right)\right\} \qquad \zeta = re^{2i\theta}}$$

(Complex squeezing parameter)

General Bogoliubov Transformation

$$\hat S^\dagger(\zeta)\,\hat a\,\hat S(\zeta) = \hat R^\dagger(\theta)\,\hat S^\dagger(r)\,\underbrace{\hat R(\theta)\,\hat a\,\hat R^\dagger(\theta)}_{e^{i\theta}\hat a}\,\hat S(r)\,\hat R(\theta)$$

$$= e^{i\theta}\,\hat R^\dagger(\theta)\left(c\hat a - s\hat a^\dagger\right)\hat R(\theta)$$

$$\Rightarrow \boxed{\begin{aligned} \hat S^\dagger(\zeta)\,\hat a\,\hat S(\zeta) &= \cosh r\,\hat a - e^{2i\theta}\sinh r\,\hat a^\dagger \\ \hat S^\dagger(\zeta)\,\hat a^\dagger\hat S(\zeta) &= \cosh r\,\hat a^\dagger - e^{-2i\theta}\sinh r\,\hat a \end{aligned}}$$

Fluctuations in $\hat X_\phi$ and $\hat P_\phi$

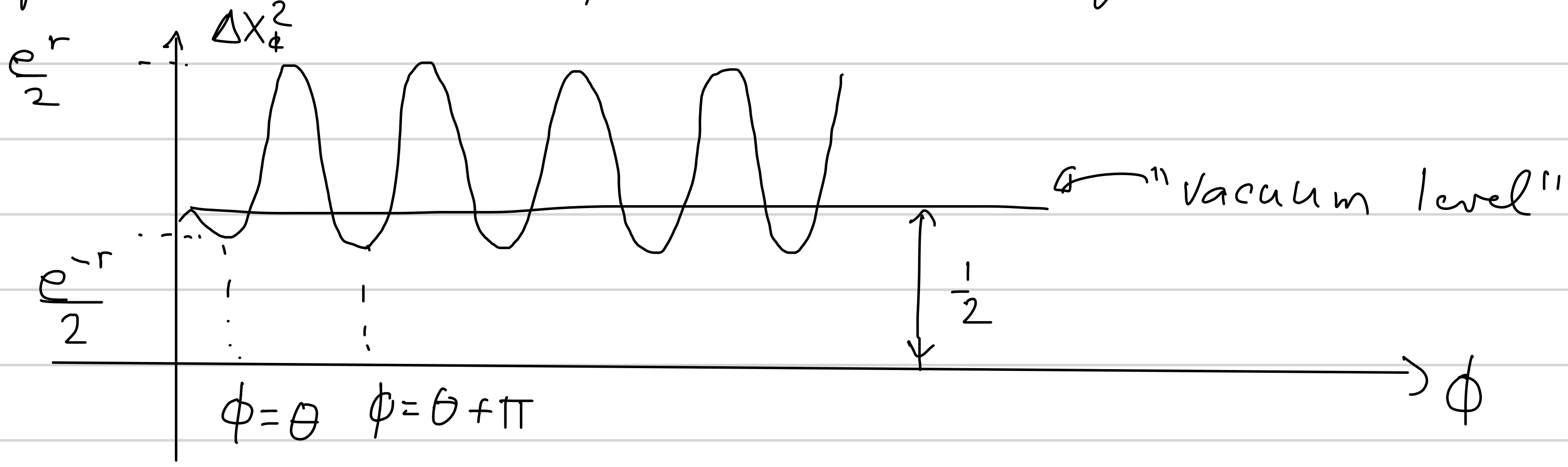
$$\langle 0_\zeta|\Delta\hat X_\phi^2|0_\zeta\rangle = \langle 0_\zeta|\hat X_\phi^2|0_\zeta\rangle = \langle 0_\zeta|\left(\hat R^\dagger(\phi)\,\hat X\,\hat R(\phi)\right)^2|0_\zeta\rangle$$

$$= \frac{1}{2}\langle 0_\zeta|\left(\hat a e^{-i\phi} + \hat a^\dagger e^{i\phi}\right)^2|0_\zeta\rangle$$

After Some Algebra

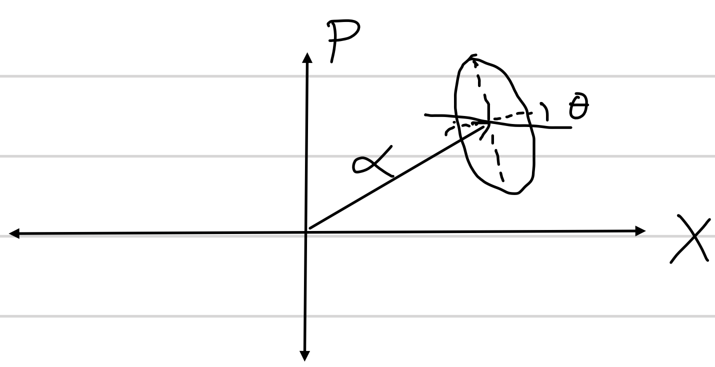
$$\Rightarrow \boxed{\langle 0_\xi | \Delta\hat{X}^2_\phi | 0_\xi \rangle = \frac{1}{2}\left[e^{-2r}\cos^2(\theta-\phi) + e^{+2r}\sin^2(\theta-\phi)\right]}$$

Thus, the squeezed vacuum shows a periodic variation in the quadrature fluctuation, as a function of ϕ



<u>Squeezed coherent state</u>

$$|\alpha,\xi\rangle = \hat{D}(\alpha)|\xi\rangle = \hat{D}(\alpha)\,\hat{S}(\xi)\,|0\rangle$$



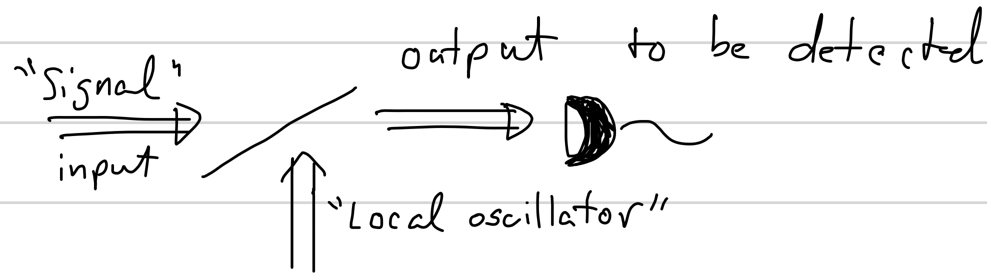
$\langle \Delta X^2 \rangle$ and $\langle \Delta P^2 \rangle$ the same as of the squeezed vacuum

<u>Note</u>: $\hat{D}(\alpha)\hat{S}(\xi) \neq \hat{S}(\xi)\hat{D}(\alpha)$. We define the state by squeezing first, then displacing. Then

$$\langle \alpha,\xi | \hat{X} | \alpha,\xi \rangle = \sqrt{2}\,\mathrm{Re}(\alpha), \quad \langle \alpha,\xi | \hat{P} | \alpha,\xi \rangle = \sqrt{2}\,\mathrm{Im}(\alpha)$$

Homodyne detection

In order to detect the phase sensitive noise associated with a squeezed state, we need a phase sensitive detection system. The basic idea is to use interference.



Classically, we take the signal to be quasimonochromatic, $\mathrm{Re}(\mathcal{E}_{in} e^{-i\omega t})$. The local oscillator is a strong field, with a stable intensity, oscillating at frequency ω_{LO}, $\mathrm{Re}(\mathcal{E}_{LO} e^{-i\omega_L t})$. The out-field is the beat-note between these to fields. If $\omega_{LO} = \omega$, this detection scheme is known as "homodyne detection." Otherwise, if $\omega \neq \omega_L$ we refer to the dection scheme of "heterodyne." We consider homodyne detection here and return to heterodyne later.

The output complex amplitude is $\mathcal{E}_{out} = t\,\mathcal{E}_{in} + r\,\mathcal{E}_{LO}$, where

t = transmission coeff, r = reflection coefficient

For a lossless, beam-splitter, by unitarity $|t|^2 + |r|^2 = 1$,

If symmetric, then $tr^* + t^* r = 0 \Rightarrow tr^* = i|t||r|$

$$\Rightarrow I_{out} = |\mathcal{E}_{out}|^2 = |t|^2 I_{in} + |r|^2 I_{LO} + \underbrace{\left(t^* r\, \mathcal{E}_{in}^* \mathcal{E}_{LO} + t r^* \mathcal{E}_{in} \mathcal{E}_{LO}^*\right)}_{\text{interference}}$$

$$\Rightarrow I_{out} = T I_{in} + R I_{LO} + \sqrt{TR}\, |\mathcal{E}_L| \left(\mathcal{E}_{in} e^{-i(\phi_{LO} + \frac{\pi}{2})} + \mathcal{E}_{in}^* e^{+i(\phi_{LO} + \frac{\pi}{2})} \right)$$

where $T = |t|^2$, $R = |r|^2$

The out signal depends on the quadrature

$$X_{in}^{\phi + \frac{\pi}{2}} = \frac{\mathcal{E}_{in} e^{-i(\phi_{LO} + \frac{\pi}{2})} + \mathcal{E}_{in}^* e^{i(\phi_{LO} + \frac{\pi}{2})}}{\sqrt{2}}$$

Ordinary homodyne: We assume the input is a weak signal. In order to not degrade the signal-to-noise, we take the beam splitter to be almost perfectly transmitting, $T \gg R$. Furthermore, we take the local oscillator to have an overcompensating, large intensity

$$R I_{LO} \gg T I_{in}$$

$$\Rightarrow I_{out} \approx \underbrace{R I_{LO}}_{\text{fixed background}} + \underbrace{\sqrt{2TR}\sqrt{I_{LO}}\, X_{in}(\phi_{LO} + \tfrac{\pi}{2})}_{\text{phase dependent quadrature}}$$

The output has a constant background, plus the phase-dependent quadrature, depending on the phase of the local oscillator.

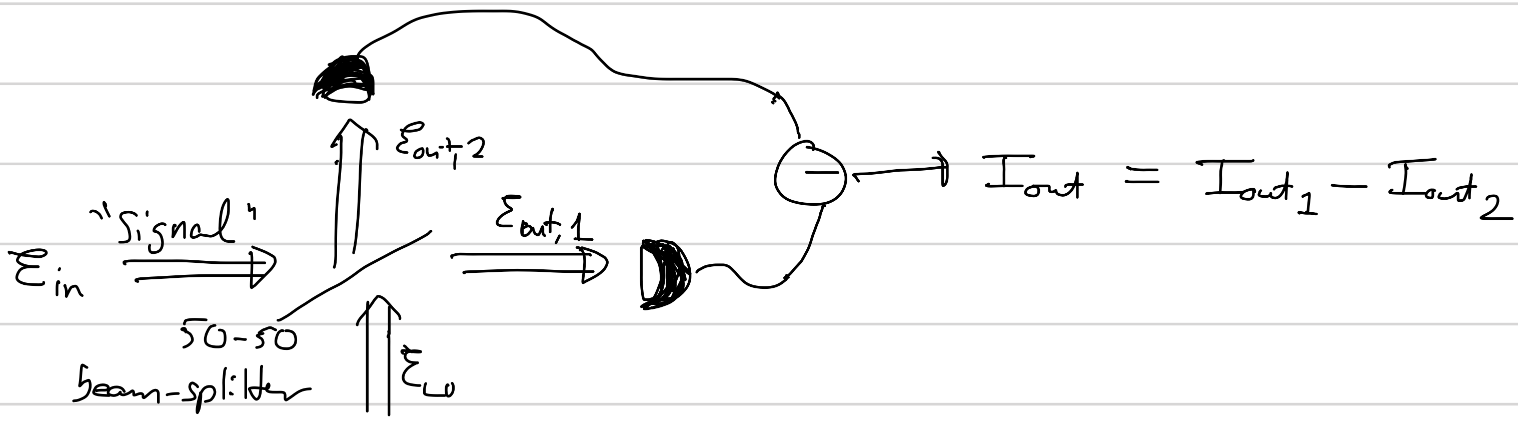
We are interested in measuring the phase-dependent noise on the signal. If the noise in the local oscillator is negligible, such that $R^2 \Delta I_{LO}^2 \ll T R I_{LO} \Delta X_{in}^2$

$$\Delta I_{out}^2 \approx T R I_L 2 \Delta X_{in}^2 (\phi_{LO} + \tfrac{\pi}{2})$$

The quantum noise seen in the ordinary homodyne detection scheme is thus the fluctuation in the phase quadrature $\Delta X_{in}^2(\phi_{LO} + \frac{\pi}{2})$.

Ordinary homodyne detection is not a good strategy when the local oscillator is too noisy due to classical statistical noise in I_L. In that case, noise reflected from the local oscillator will add to the noise in the phase quadrature of the signal that we are trying to measure.

A more robust scheme is to cancel the effect of noise seen in the local oscillator using a balanced homodyne detector.



We now detect the field at the two output ports of a 50-50 beam splitter after interfering the signal with the local oscillator. Again assuming a symmetric beam-splitter

$$\mathcal{E}_{1,out} = \frac{1}{\sqrt{2}}\left(\mathcal{E}_{in} + i\,\mathcal{E}_{LO}\right)$$
$$\mathcal{E}_{2,out} = \frac{1}{\sqrt{2}}\left(i\,\mathcal{E}_{in} + \mathcal{E}_{LO}\right)$$

$$\Rightarrow I_{1,out} = \frac{1}{2}\left(I_{in} + I_{LO} + \sqrt{2I_{LO}}\,X_{in}\left(\phi_{LO}+\frac{\pi}{2}\right)\right)$$
$$I_{2,out} = \frac{1}{2}\left(I_{in} + I_{L} - \sqrt{2I_{LO}}\,X_{in}\left(\phi_{LO}+\frac{\pi}{2}\right)\right)$$

$$\Rightarrow \boxed{I_{1,out} - I_{2,out} = \sqrt{2I_{LO}}\,X_{in}\left(\phi_{LO}+\frac{\pi}{2}\right)}$$

The balanced homodyne detector directly measures the phase quadrature of $X_{in}(\phi_{LO}+\frac{\pi}{2})$ of the input field, and the fluctuation $\Delta I_{1,out}^2 - \Delta I_{2,out}^2 = 2 I_{LO}\,\Delta X_{in}^2(\phi_{LO}+\frac{\pi}{2})$ are due solely to the fluctuations in the signal quadrature, and not to any noise in the local oscillator.

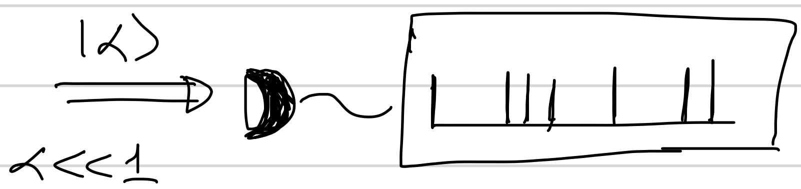
Quantum theory – Shot Noise

The fundamendal fluctuations in photocounts arising from the quantum uncertainty is known as shot-noise. Assuming an ideal photodector with unit detection efficiency (often known as the detector's "quantum efficiency"), the fluctuations in photo-counts in a given time interval is due to the quantum uncertainty of the photon number entering the detector.

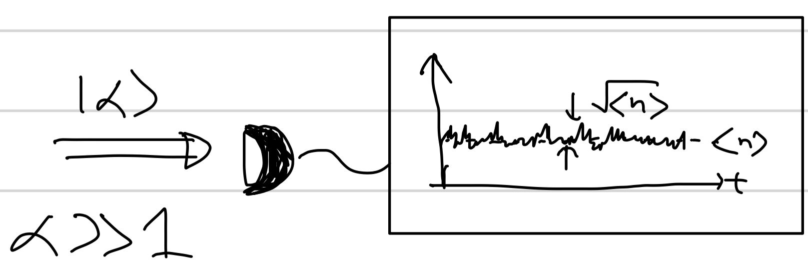
For a coherent state, the number fluctuations are Poissonian

$$\Delta n = \sqrt{\langle n \rangle}$$

When $\langle n \rangle \ll 1$, within the given detector response time, we are in "Geiger mode", and we expect a discrete set of photon counts



When $\langle n \rangle \gg 1$, we expect to approach a continuous photocurrent



The continuous shot noise can be seen as the limit of Poisson fluctuations

$$\lim_{\langle n \rangle \to \infty} \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} \rightarrow \frac{1}{\sqrt{2\pi \langle n \rangle}} e^{-\frac{(n - \langle n \rangle)^2}{2\langle n \rangle}}$$

Gaussian
$\bar{n} = \langle n \rangle$
$\Delta n = \sqrt{\langle n \rangle}$

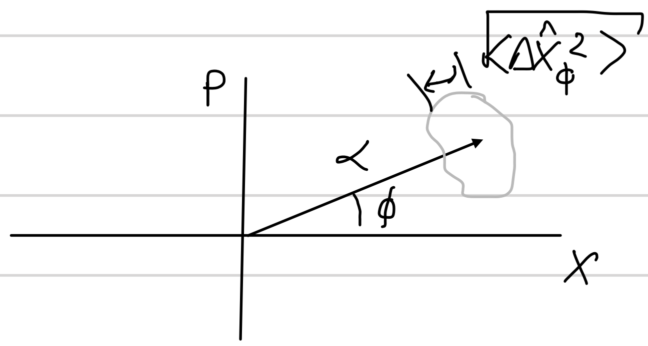
The shot-noise fluctuations around the mean are, to good approximation, Gaussian distributed → "Gaussian white noise"

Another way to see the continuum limit is to make the "Mollow transformation" and write the mode operator in terms of mean field plus fluctuation

$$\hat{b} \equiv \hat{D}^\dagger \hat{a} \hat{D} = \underset{\substack{\uparrow \\ \text{mean field}}}{\alpha} + \underbrace{\hat{a}}_{\text{fluctuation}} \qquad |\psi\rangle \Rightarrow \hat{D}^\dagger(\alpha)|\psi\rangle \equiv |\tilde{\psi}\rangle \; \left(= |0\rangle \text{ if } |\psi\rangle = |\alpha\rangle \right)$$

$$\hat{n} = \hat{b}^\dagger \hat{b} = |\alpha|^2 + \alpha^* \hat{a} + \alpha \hat{a}^\dagger + \hat{a}^\dagger \hat{a} \underset{\alpha \gg 1}{\approx} |\alpha|^2 + \sqrt{2|\alpha|^2}\, \hat{X}_\phi$$

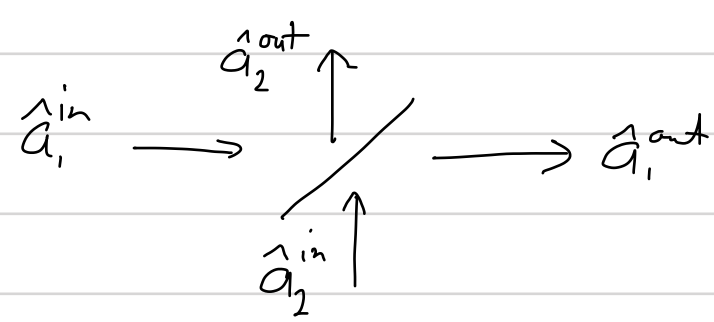
where $\hat{X}_\phi = \dfrac{\hat{a} e^{-i\phi} + \hat{a}^\dagger e^{i\phi}}{\sqrt{2}}$ is the phase quadrature along the phase of the mean field (amplitude quadrature).



In this approximation $\Delta n^2 = 2|\alpha|^2 \langle \Delta \hat{X}_\phi^2 \rangle$. For a coherent state, $\langle \Delta \hat{X}_\phi^2 \rangle = \frac{1}{2}$ and $\Delta n^2 = |\alpha|^2 = \langle n \rangle$, as expected. The operator $\hat{n}$ then represents a Gaussian random variable.

Quantum theory of homodyne detection

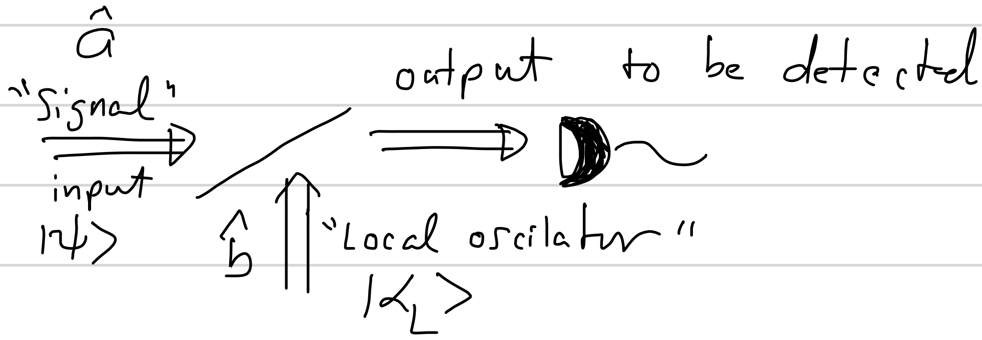
The linear transformation of the beam splitter of the classical planewave mode amplitudes translates into a unitary transformation on the mode creation/annihilation operators



$$\hat a_i^{out} = \underset{\text{scattering operator}}{\hat U^\dagger} \hat a_i^{in} \hat U = \sum_j{}' \underset{\text{linear mode transformation}}{\Lambda_{ij}} \hat a_j^{in}$$

$$\Rightarrow \hat a_1^{out} = t\hat a_1^{in} + r\hat a_2^{in}, \qquad \hat a_2^{out} = t\hat a_2^{in} + r\hat a_1^{in}$$

Ordinary homodyne:



$$\hat a_{out} = t\hat a_{in} + r\hat b_{in}$$

$$\Rightarrow \hat n_{a_{out}} = |t|^2 \hat n_{a_{in}} + |r|^2 \hat n_{b_{in}} + t r^* \hat a_{in} \hat b_{in}^\dagger + t^* r \hat a_{in}^\dagger \hat b_{in}$$

$$= T\hat n_{a_{in}} + R\hat n_{b_{in}} + \sqrt{2TR}\left(\frac{-i\hat a_{in}\hat b_{in}^\dagger + i\hat a_{in}^\dagger \hat b_{in}}{\sqrt{2}}\right)$$

$$\langle \hat n_{a_{out}}\rangle = \langle\psi,\alpha_L|\, \hat n_{a_{out}}\, |\psi,\alpha_L\rangle$$

$$= \underbrace{\cancel{T\langle\psi|\hat n_{a_{in}}|\psi\rangle}}_{\text{neglect}} + R|\alpha_L|^2 + \sqrt{2TR|\alpha_L|^2}\,\langle\psi|\hat X_{in}(\phi_L+\tfrac{\pi}{2})|\psi\rangle$$

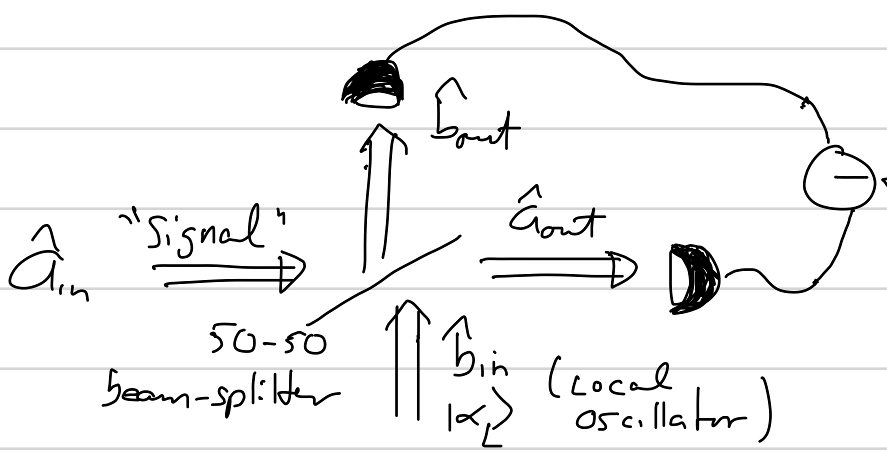
$$\hat X_{in}(\phi_L+\tfrac{\pi}{2}) = \frac{\hat a_{in} e^{-i(\phi+\frac{\pi}{2})} + \hat a_{in}^\dagger e^{i(\phi+\frac{\pi}{2})}}{\sqrt{2}}$$

$$\langle\Delta\hat n^2_{a_{out}}\rangle = \underbrace{R^2\langle\Delta\hat n^2_{b_{in}}\rangle}_{\substack{\text{Reflected noise of LO}\\ =|\alpha_L|^2 \text{ for coherent state}}} + TR|\alpha_L|^2\, 2\underbrace{\langle\Delta\hat X^2_{in}(\phi_L+\tfrac{\pi}{2})\rangle}_{\text{Quadrature fluct}}$$

When $T > R$ $\quad \langle\Delta\hat n^2_{a_{out}}\rangle \simeq TR|\alpha_L|^2\, 2\langle\Delta\hat X^2_{in}(\phi_L+\tfrac{\pi}{2})\rangle$

Measure quadrature fluctuations

When measuring quantum noise, it is particularly important that we remove any excess noise arising from the local oscillator. The most sensitive measurement is then done by balanced homodyne



$$\hat{I}_{out} = \hat{n}_{a_{out}} - \hat{n}_{b_{out}}$$

$$\hat{a}_{out} = \frac{1}{\sqrt{2}}(\hat{a}_{in} + i\hat{b}_{in}), \qquad \hat{b}_{out} = \frac{1}{\sqrt{2}}(\hat{b}_{in} + i\hat{a}_{in})$$

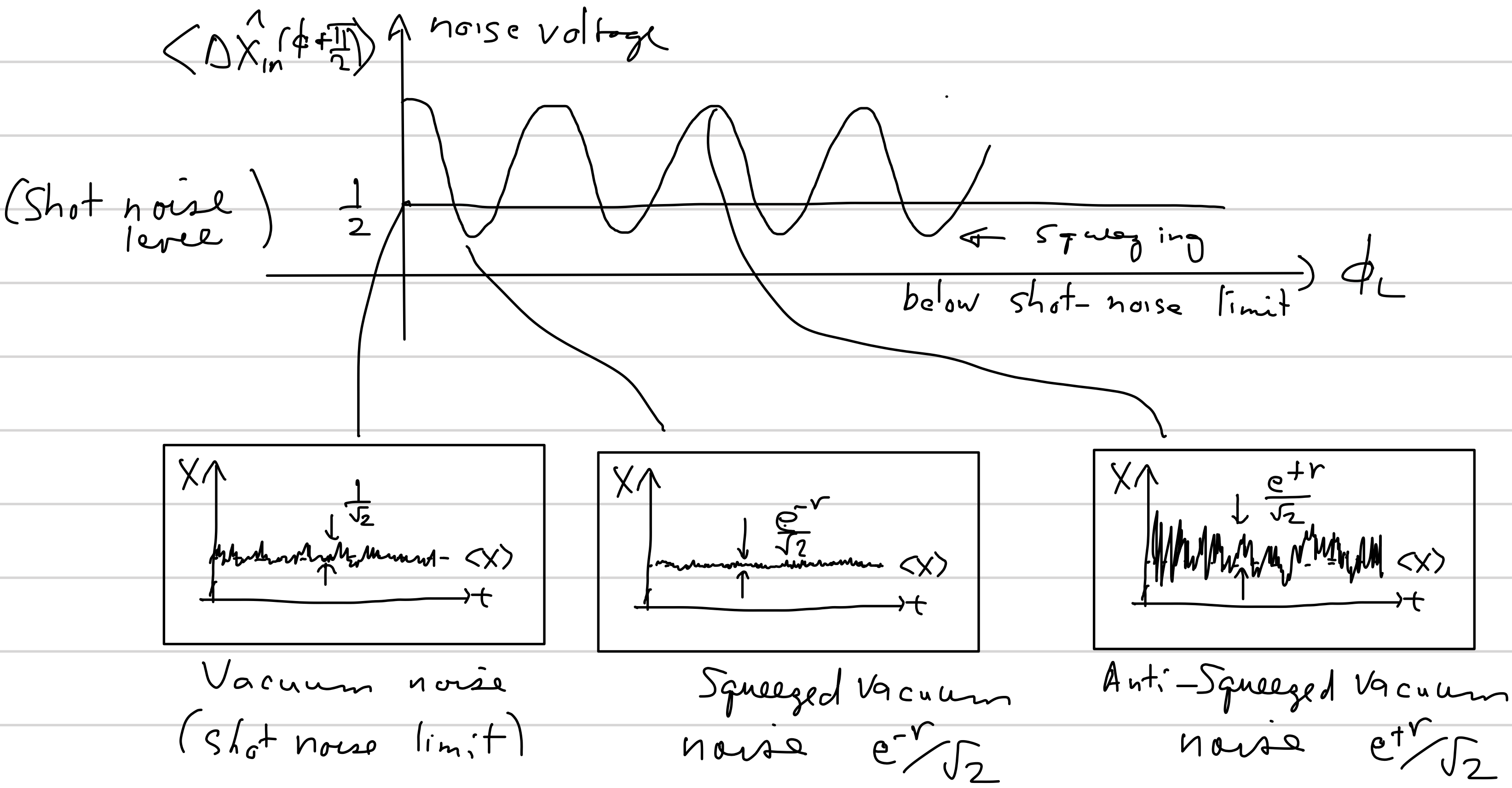
$$\Rightarrow \hat{I}_{out} = \hat{n}_{a_{out}} - \hat{n}_{b_{out}} = \sqrt{2}\left(\frac{-i\hat{a}_{in}\hat{b}_{in}^{\dagger} + i\hat{a}_{in}^{\dagger}\hat{b}_{in}}{\sqrt{2}}\right)$$

Making the Mollow transformation $\hat{b}_{in} = \underbrace{|\alpha_L| e^{i\phi_L}}_{\text{mean field}} + \underbrace{\Delta\hat{b}_{in}}_{\text{fluctuating}}$

$$\boxed{\begin{array}{l} \Rightarrow \hat{I}_{out} \approx |\alpha_L| \sqrt{2}\, \hat{X}_{in}(\phi_L + \frac{\pi}{2}) \quad \text{when } |\alpha_L|^2 >> \langle \Delta\hat{b}^2 \rangle \\ \Rightarrow \text{Balanced homodyne detector measures quadrature} \end{array}}$$

$$\langle \Delta\hat{I}_{out}^2 \rangle = |\alpha_L|^2 \, 2\langle \Delta\hat{X}_{in}^2(\phi_L + \tfrac{\pi}{2}) \rangle$$

Given a squeezed state we can measure the phase-dependent fluctuations in the quadratures as a function of ϕ_L



The level of fluctuations seen in the photodetector depends on the phase of local oscillator. Interfering the squeezed vacuum with the coherent state of the local oscillator has reduce the fluctions beyond the vacuum level, sometimes known as the "<u>standard quantum limit</u>".